Chapter 5 • Applications of Derivatives

Review of Prerequisite Skills

5. **a.** $3(x-2)+2(x-1)-6=0$

$$3x-6+2x-2-6=0$$
$$5x=14$$
$$x=\frac{14}{5}$$

e. $\frac{6}{t}+\frac{t}{2}=4$

$$12+t^2=8t$$
$$t^2-8t+12=0$$
$$(t-6)(t-2)=0$$
$$\therefore\ t=2 \quad \text{or} \quad t=6$$

f. $x^3+2x^2-3x=0$

$$x(x^2+2x-3)=0$$
$$x(x+3)(x-1)=0$$
$$x=0 \quad \text{or} \quad x=-3 \quad \text{or} \quad x=1$$

g. $x^3-8x^2+16x=0$

$$x(x^2-8x+16)=0$$
$$x(x-4)^2=0$$
$$x=0 \quad \text{or} \quad x=4$$

h. $4t^3+12t^2-t-3=0$

$$4t^2(t+3)-1(t+3)=0$$
$$(t+3)(4t^2-1)=0$$
$$(t+3)(2t-1)(2t+1)=0$$
$$t=-3 \quad \text{or} \quad t=\frac{1}{2} \quad \text{or} \quad t=-\frac{1}{2}$$

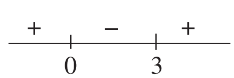
i. $4t^4-13t^2+9=0$

$$(4t^2-9)(t^2-1)=0$$
$$t=\pm\frac{9}{4} \quad \text{or} \quad t=\pm 1$$

6. **a.** $3x-2>7$

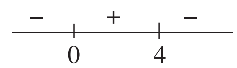
$$3x>9$$
$$x>3$$

b. $x(x-3)>0$



$$x<0 \quad \text{or} \quad x>3$$

c. $-x^2+4x>0$



$$x(x-4)<0$$
$$0<x<4$$

Exercise 5.1

2. **d.** $3xy^2+y^3=8$

$$3y^2+3x2y\frac{dy}{dx}+3y^2\frac{dy}{dx}=0$$
$$\frac{dy}{dx}(2xy+y^2)=-y^2$$
$$\frac{dy}{dx}=\frac{-y^2}{2xy+y^2}$$

f. $9x^2-16y^2=-144$

$$18x-32y\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=\frac{9x}{16y}$$

g. $-\frac{x^2}{16}+\frac{3}{13}y^2=1$

$$\frac{2x}{16}+\frac{6}{13}y\frac{dy}{dx}=0$$
$$26x+96y\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{13x}{48y}$$

h. $3x^2+4xy^3=9$

$$6x+4y^3+4x3y^2\frac{dy}{dx}=0$$
$$6xy^2\frac{dy}{dx}=-3x-2y^3$$
$$\frac{dy}{dx}=\frac{-3x-2y^3}{6xy^2}$$

j. $x^3 + y^3 = 6xy$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + \frac{dy}{dx}(6x)$$

$$(3y^2 - 6x)\frac{dy}{dx} = -3x^2 + 6y$$

$$\frac{dy}{dx} = \frac{-3x^2 + 6y}{3y^2 - 6x}$$

$$= \frac{-x^2 + 2y}{y^2 - 2x}$$

k. $x^3y^3 = 144$

$$3x^2y^3 + 3y^2\frac{dy}{dx}x^3 = 0$$

$$\frac{dy}{dx} = -\frac{x^2y^3}{x^3y^2}$$

$$= -\frac{y}{x}$$

m. $xy^3 - x^3y = 2$

$$y^3 + 3y^2\frac{dy}{dx}x - \left[3x^2y + \frac{dy}{dx}x^3\right] = 0$$

$$(3y^2 - x^3)\frac{dy}{dx} = 3x^2y - y^3$$

$$\frac{dy}{dx} = \frac{3x^2y - y^3}{3y^2 - x^3}$$

n. $\sqrt{x} + \sqrt{y} = 5$

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5$$

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}$$

$$= -\frac{\sqrt{y}}{\sqrt{x}}$$

o. $(x + y)^2 = x^2 + y^2$

$$2(x + y)\left[1 + \frac{dy}{dx}\right] = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}[x + y - y] = x - x - y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

3. a. $x^2 + y^2 = 13$

$$2x + 2y\frac{dy}{dx} = 0$$

At $(2, -3)$,

$$2(2) + 2(-3)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2}{3}.$$

The equation of the tangent at $(2, -3)$ is

$$y = \frac{2}{3}x + b.$$

At $(2, -3)$,

$$-3 = \frac{2}{3}(2) + b$$

$$-9 = 4 + 3b$$

$$-13 = 3b$$

$$-\frac{13}{3} = b.$$

Therefore, the equation of the tangent to $x^2 + y^2 = 13$ is $y = \frac{2}{3} \times -\frac{13}{3}$.

c. $\frac{x^2}{25} - \frac{y^2}{36} = -1$

$$\frac{2x}{25} - \frac{2y}{36}\frac{dy}{dx} = 0$$

$$36x - 25y\frac{dy}{dx} = 0$$

At $(5\sqrt{3}, -12)$,

$$180\sqrt{3} + 300\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3\sqrt{3}}{5}.$$

The equation of the tangent is $y = mx + b$.

At $(5\sqrt{3}, -12)$ and with $m = -\frac{3\sqrt{3}}{5}$,

$$-12 = -\frac{3\sqrt{3}}{5}\left(5\sqrt{3}\right) + b$$

$$-12 = -9 + b$$
$$-3 = b$$

Therefore, the equation of the tangent is

$$y = -\frac{3\sqrt{3}}{5}x - 3.$$

4. $x + y^2 = 1$

The line $x + 2y = 0$ has slope of $-\frac{1}{2}$.

$$1 + 2y\frac{dy}{dx} = 0$$

Since the tangent line is parallel to $x + 2y = 0$,

then $\frac{dy}{dx} = -\frac{1}{2}$.

$$\therefore\ 1 + 2y\left(-\frac{1}{2}\right) = 0$$
$$1 - y = 0$$
$$y = 1$$

Substituting,

$x + 1 = 1$

$x = 0$

Therefore, the tangent line to the curve $x + y^2 = 1$ is parallel to the line $x + 2y = 0$ at $(0, 1)$.

5. **a.** $5x^2 - 6xy + 5y^2 = 16$

$$10x - \left(6y + \frac{dy}{dx}(6x)\right) + 10y\frac{dy}{dx} = 0 \ (1)$$

At $(1, -1)$,

$$10 - \left(-6 + 6\frac{dy}{dx}\right) - 10\frac{dy}{dx} = 0$$

$$16 - 16\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 1.$$

b. When the tangent line is horizontal, $\frac{dy}{dx} = 0$.

Substituting,

$10x - (6y + 0) + 0 = 0$.

$y = \frac{5}{3}x$ at the point (x_1, y_1) of tangency:

substitute $y_1 = \frac{5}{3}x_1$ into $5x_1^2 - 6x_1y_1 + 5y_1^2 = 16$.

$$5x_1^2 - 6x_1\left(\frac{5}{3}x_1\right) + 5\left(\frac{25}{9}x_1^2\right) = 16$$

$$45x_1^2 - 90x_1^2 + 125x_1^2 = 144$$

$$80x_1^2 = 144$$

$$5x_1^2 = 9$$

$x_1 = \frac{3}{\sqrt{5}}$ or $x_1 = -\frac{3}{\sqrt{5}}$

$y_1 = \frac{5}{\sqrt{5}}$ or $y_1 = \sqrt{5}$

$y_1 = -\frac{5}{\sqrt{5}}$ or $y_1 = -\sqrt{5}$

Therefore, the required points are $\left(\frac{3}{\sqrt{5}}, \sqrt{5}\right)$ and $\left(-\frac{3}{\sqrt{5}}, -\sqrt{5}\right)$.

7. $x^3 + y^3 - 3xy = 17$

$$3x^2 + 3y^2\frac{dy}{dx} - \left[3y + \frac{dy}{dx}(3x)\right] = 0$$

At $(2, 3)$,

$$12 + 27\frac{dy}{dx} - 9 - 6\frac{dy}{dx} = 0$$

$$21\frac{dy}{dx} = -3.$$

The slope of the tangent is $\frac{dy}{dx} = -\frac{1}{7}$.

Therefore, the slope of the normal at $(2, 3)$ is 7.

The equation of the normal at $(2, 3)$ is $\frac{y - 3}{x - 2} = 7$

$y - 3 = 7x - 14$ or $7x - y - 11 = 0$

9. $4x^2y - 3y = x^3$

a. $8xy + \frac{dy}{dx}(4x^2) - 3\frac{dy}{dx} = 3x^2$

$$\frac{dy}{dx}(4x^2 - 3) = 3x^2 - 8xy$$

$$\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3} \qquad (1)$$

b. $y(4x^2 - 3) = x^3$

$$y = \frac{x^3}{4x^2 - 3}$$

$$\frac{dy}{dx} = \frac{3x^2(4x^2 - 3) - 8x(x^3)}{(4x^2 - 3)^2}$$

$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2 - 3)^2}$$

$$= \frac{4x^4 - 9x^2}{(4x^2 - 3)^2} \qquad (2)$$

We must show that (1) is equivalent to (2).

From (1): $\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}$

and substituting, $y = \frac{x^3}{4x^2 - 3}$

$$\frac{dy}{dx} = \frac{3x^2 - 8x\left(\frac{x^3}{4x^2 - 3}\right)}{4x^2 - 3}$$

$$= \frac{3x^2(4x^2 - 3) - 8x^4}{(4x^2 - 3)^2}$$

$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2 - 3)^2}$$

$$= \frac{4x^4 - 9x^2}{(4x^2 - 3)^2} = (2), \text{ as required.}$$

11. $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 10,\ x \neq\ y \neq\ 0,\ \frac{dy}{dx} = \frac{y}{x}$

$$\frac{1}{2}\left(\frac{x}{y}\right)^{-\frac{1}{2}} \frac{1y - \frac{dy}{dx}x}{y^2} + \frac{1}{2}\left(\frac{y}{x}\right)^{-\frac{1}{2}} \frac{\frac{dy}{dx}x - y}{x^2} = 0$$

$$\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \frac{1y - \frac{dy}{dx}x}{y^2} + \frac{x^{\frac{1}{2}}}{y^{-\frac{1}{2}}} \frac{\frac{dy}{dx}x - y}{x^2} = 0$$

Multiply by x^2y^2:

$$x^{\frac{3}{2}}y^{\frac{1}{2}}\left(y - x\frac{dy}{dx}\right) + 2^{\frac{1}{2}}y^{\frac{3}{2}}\left(\frac{dy}{dx}x - y\right) = 0$$

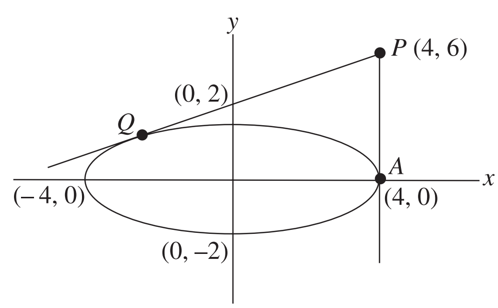
$$x^{\frac{3}{2}}y^{\frac{3}{2}} - x^{\frac{5}{2}}y^{\frac{1}{2}}\frac{dy}{dx} + x^{\frac{3}{2}}y^{\frac{3}{2}}\frac{dy}{dx} - x^{\frac{1}{2}}y^{\frac{5}{2}} = 0$$

$$\frac{dy}{dx}\left(x^{\frac{3}{2}}y^{\frac{3}{2}} - x^{\frac{5}{2}}y^{\frac{1}{2}}\right) = x^{\frac{1}{2}}y^{\frac{5}{2}} - x^{\frac{3}{2}}y^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}}y^{\frac{3}{2}}(y - x)}{x^{\frac{3}{2}}y^{\frac{1}{2}}(y - x)}$$

$$\frac{dy}{dx} = \frac{y}{x}, \text{ as required.}$$

12.



Let Q have coordinates

$(q, f(q)) = \left(q, \frac{\sqrt{16 - q^2}}{2}\right),\ q < 0.$

For $x^2 + 4y^2 = 16$

$$2x + 3y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{4y}.$$

$$\text{At } Q,\ \frac{dy}{dx} = -\frac{q}{2\sqrt{16 - q^2}}$$

The line through P has equation $\frac{y-6}{x-4} = m$.

Since PQ is the slope of the tangent line to $x^2 + 4y^2 = 16$, we conclude:

$m = \frac{dy}{dx}$ at point Q.

$$\therefore \frac{\frac{\sqrt{16-q^2}}{2} - 6}{q-4} = -\frac{q}{2\sqrt{16-q^2}}$$

$$\frac{\sqrt{16-q^2} - 12}{2(q-4)} = -\frac{9}{2\sqrt{16-q^2}}$$

$$16 - q^2 - 12\sqrt{16-q^2} = -q(q-4)$$

$$16 - q^2 - 12\sqrt{16-q^2} = -q^2 + 4q$$

$$4 - q = 3\sqrt{16-q^2}$$

$$\begin{aligned} 16 - 8q + q^2 &= 9(16 - q^2) \\ 16 - 8q + q^2 &= 144 - 9q^2 \\ 10q^2 - 8q - 128 &= 0 \\ 5q^2 - 4q - 64 &= 0 \\ (5q + 16)(q - 4) &= 0 \end{aligned}$$

$q = -\frac{16}{5}$ or $q = 4$ (as expected; see graph)

$f(q) = \frac{6}{5}$ or $f(q) = 0$

$$\frac{dy}{dx} = \frac{\frac{16}{5}}{4\left(\frac{6}{5}\right)} \quad \text{or} \quad f(q) = 0$$

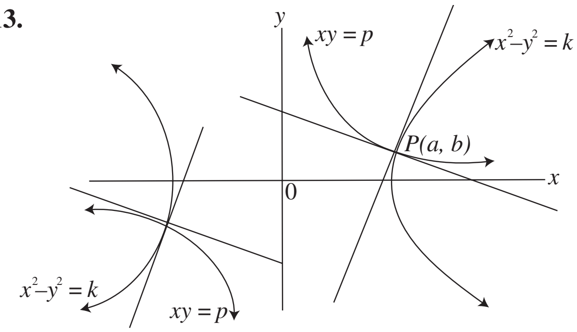
$$= \frac{2}{3}$$

Equation of the tangent at Q is

$\frac{y-6}{x-4} = \frac{2}{3}$ or $2x - 3y + 10 = 0$

or equation of tangent at A is $x = 4$.

13.



Let $P(a, b)$ be the point of intersection where $a \neq 0$ and $b \neq 0$.

For $x^2 - y^2 = k$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

At $P(a, b)$,

$$\frac{dy}{dx} = \frac{a}{b}.$$

For $xy = P$,

$$1 \bullet y + \frac{dy}{dx}x = P$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

At $P(a, b)$,

$$\frac{dy}{dx} = -\frac{b}{a}.$$

At point $P(a, b)$, the slope of the tangent line of $xy = P$ is the negative reciprocal of the slope of the tangent line of $x^2 - y^2 = k$. Therefore, the tangent lines intersect at right angles, and thus, the two curves intersect orthogonally for all values of the constants k and P.

14. $\sqrt{x} + \sqrt{y} = \sqrt{k}$

diff wrt x

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Let $P(a, b)$ be the point of tangency.

$$\therefore \frac{dy}{dx} -\frac{\sqrt{b}}{\sqrt{a}}$$

Equation of tangent line l at P is

$$\frac{y - b}{x - a} = -\frac{\sqrt{b}}{\sqrt{a}}.$$

x – intercept is found when $y = 0$.

$$\therefore \frac{-b}{x - a} = -\frac{\sqrt{b}}{\sqrt{a}}$$

$$-b\sqrt{a} = -\sqrt{b}\,x + a\sqrt{b}$$

$$x = \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$$

Therefore, the x–intercept is $\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$.

For the y–intercept, let $x = 0$,

$$\frac{y - b}{-a} = -\frac{\sqrt{b}}{\sqrt{a}}.$$

y – intercept is $\frac{a\sqrt{b}}{\sqrt{a}} + b$.

The sum of the intercepts is

$$\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}} + \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{a}}$$

$$= \frac{a^{\frac{3}{2}}b^{\frac{1}{2}} + 2ab + b^{\frac{3}{2}}a^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$

$$= \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}(a + 2\sqrt{a}\sqrt{b} + b)}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$

$$= a + 2\sqrt{a}\sqrt{b} + b$$

$$= \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2$$

Since $P(a, b)$ is on the curve, then $\sqrt{a} + \sqrt{b} = \sqrt{k}$,

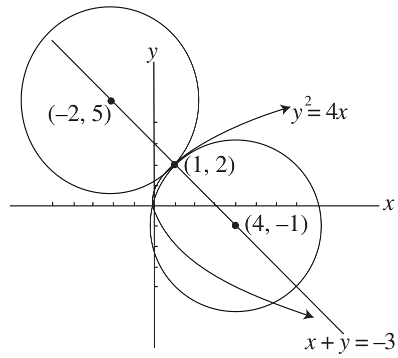
or $a^{\frac{1}{2}} + b^{\frac{1}{2}} = k^{\frac{1}{2}}$.

Therefore, the sum of the intercepts

$$= \left(k^{\frac{1}{2}}\right)^2$$

$= k$, as required.

15.



$y^2 = 4x$

$$2y\frac{dy}{dx} = 4$$

At $(1, 2)$, $\frac{dy}{dx} = 1$.

Therefore, the slope of the tangent line at $(1, 2)$ is 1 and the equation of the normal is

$$\frac{y - 2}{x - 1} = -1 \quad \text{or} \quad x + y = 3.$$

The centres of the two circles lie on the straight line $x + y = 3$. Let the coordinates of the centre of each circle be $(p, q) = (p, 3 - p)$. The radius of each circle is $3\sqrt{2}$. Since $(1, 2)$ is on the circumference of the circles,

$$(p - 1)^2 + (3 - p - 2)^2 = r^2$$
$$p^2 - 2p + 1 + 1 - 2p + p^2 = (3\sqrt{2})^2$$
$$2p^2 - 4p + 2 = 18$$
$$p^2 - 2p - 8 = 0$$
$$(p - 4)(p + 2) = 0$$
$$p = 4 \quad \text{or} \quad p = -2$$
$$\therefore q = -1 \quad \text{or} \quad q = 5.$$

Therefore, the centres of the circles are $(-2, 5)$ and $(4, -1)$. The equations of the circles are

$$(x + 2)^2 + (y - 5)^2 = 18$$

and $(x - 4)^2 + (y + 1)^2 = 18$.

Exercise 5.2

3. **a.** $s(t) = 5t^2 - 3t + 15$
$v(t) = 10t - 3$
$a(t) = 10$

b. $s(t) = 2t^3 + 36t - 10$
$v(t) = 6t^2 + 36$
$a(t) = 12t$

e. $s(t) = \sqrt{t + 1}$

$$v(t) = \frac{1}{2}(t + 1)^{-\frac{1}{2}}$$

$$a(t) = -\frac{1}{4}(t + 1)^{-\frac{3}{2}}$$

f. $s(t) = \dfrac{9t}{t + 3}$

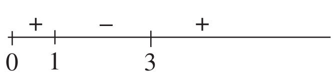
$$v(t) = \frac{9(t + 3) - 9t}{(t + 3)^2}$$

$$= \frac{27}{(t + 3)^2}$$

$$a(t) = -54(t + 3)^{-3}$$

5. $s = \dfrac{1}{3}t^3 - 2t^2 + 3t$

$v = t^2 - 4t + 3$
$a = 2t - 4$
For $v = 0$,
$(t - 3)(t - 1) = 0$
$t = 3$ or $t = 1$.



The direction of the motion of the object changes at $t = 1$ and $t = 3$.
Initial position is $s(0) = 0$.

Solving,

$$0 = \frac{1}{3}t^3 - 2t^2 + 3t$$
$$= t^3 - 6t^2 + 9t$$
$$= t(t^2 - 6t + 9)$$
$$= t(t - 3)^2$$

$\therefore\ t = 0$ or $t = 3$
$s = 0$ or $s = 0$.

The object returns to its initial position after 3 s.

6. **a.** $s = -\dfrac{1}{3}t^2 + t + 4$

$$v = -\frac{2}{3}t + 1$$

$$v(1) = -\frac{2}{3} + 1$$

$$= \frac{1}{3}$$

$$v(4) = -\frac{2}{3}(4) + 1$$

$$= -\frac{5}{3}$$

For $t = 1$, moving in a positive direction.
For $t = 4$, moving in a negative direction.

b. $s(t) = t(t - 3)^2$
$v(t) = (t - 3)^2 + 2t(t - 3)$
$= (t - 3)(t - 3 + 2t)$
$= (t - 3)(3t - 3)$
$= 3(t - 1)(t - 3)$
$v(1) = 0$
$v(4) = 9$
For $t = 1$, the object is stationary.
$t = 4$, the object is moving in a positive direction.

8. $s(t) = 40t - 5t^2$
$v(t) = 40 - 10t$

a. When $v = 0$, the object stops rising.
$\therefore t = 4$ s

b. Since $s(t)$ represents a quadratic function that opens down because $a = -5 < 0$, a maximum height is attained. It occurs when $v = 0$. Height is a maximum for
$s(4) = 160 - 5(16)$
$= 80$ m.

10. $s(t) = t^{\frac{5}{2}}(7 - t)$

a. $v(t) = \frac{5}{2}t^{\frac{3}{2}}(7 - t) - t^{\frac{5}{2}}$

$= \frac{35}{2}t^{\frac{3}{2}} - \frac{5}{2}t^{\frac{5}{2}} - t^{\frac{5}{2}}$

$= \frac{35}{2}t^{\frac{3}{2}} - \frac{7}{2}t^{\frac{5}{2}}$

b. $a(t) = \frac{105}{2}t^{\frac{1}{2}} - \frac{35}{4}t^{\frac{3}{2}}$

c. The direction of the motion changes when its velocity changes from a positive to a negative value or visa versa.

$v(t) = \frac{7}{2}t^{\frac{3}{2}}(5 - t) \therefore r(t) = 0$ for $t = 5$

t	$0 \le t < 5$	$t = 5$	$t > 5$
$\boldsymbol{v(t)}$	$(+)(+) = +$	0	$(+)(-) = -$

Therefore, the object changes direction at 5 s.

d. $a(t) = 0$ for $\frac{35}{4}t^{\frac{1}{2}}(6 - t) = 0$.

$\therefore t = 0$ or $t = 6$ s.

t	$0 < t < 6$	$t = 6$	$t > 6$
$\boldsymbol{a(t)}$	$(+)(+) = +$	0	$(+)(-) = -$

Therefore, the acceleration is positive for $0 < t < 6$ s.
Note: $t = 0$ yields $a = 0$.

e. At $t = 0$, $s(0) = 0$. Therefore, the object's original position is at 0, the origin.

When $s(t) = 0$,
$t^{\frac{5}{2}}(7 - t) = 0$
$t = 0$ or $t = 7$.
Therefore, the object is back to its original position after 7 s.

12. $s(t) = 6t^2 + 2t$
$v(t) = 12t + 2$
$a(t) = 12$

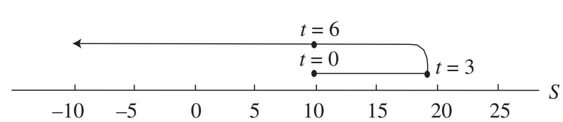
a. $v(8) = 96 + 2 = 98$ m/s
Thus, as the dragster crosses the finish line at $t = 8$ s, the velocity is 98 m/s. Its acceleration is constant throughout the run and equals 12 m/s^2.

b.
$$\begin{aligned} s &= 60 \\ 6t^2 + 2t - 60 &= 0 \\ 2(3t^2 + t - 30) &= 0 \\ 2(3t + 10)(t - 3) &= 0 \end{aligned}$$

$t = \frac{-10}{3}$ or $t = 3$

inadmissible $\quad v(3) = 36 + 2$
$0 \le t \le 8 \quad = 38$
Therefore, the dragster was moving at 38 m/s when it was 60 m down the strip.

13. a. $s = 10 + 6t - t^2$
$v = 6 - 2t$
$= 2(3 - t)$
$a = -2$
The object moves to the right from its initial position of 10 m from the origin, 0, to the 19 m mark, slowing down at a rate of 2 m/s^2. It stops at the 19 m mark then moves to the left speeding up at 2 m/s^2 as it goes on its journey into the universe. It passes the origin after $(3 + \sqrt{19})$ s.



b. $s = t^3 - 12t - 9$

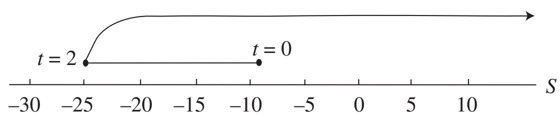
$v = 3t^2 - 12$

$= 3(t^2 - 4)$

$= 3(t - 2)(t + 2)$

$a = 6t$

The object begins at 9 m to the left of the origin, 0, and slows down to a stop after 2 s when it is 25 m to the left of the origin. Then, the object moves to the right speeding up at faster rates as time increases. It passes the origin just before 4 s (approximately 3.7915) and continues to speed up as time goes by on its journey into space.



14. $s(t) = t^5 - 10t^2$

$v(t) = 5t^4 - 20t$

$a(t) = 20t^3 - 20$

For $a(t) = 0$,

$20t^3 - 20 = 0$

$20(t^3 - 1) = 0$

$t = 1.$

Therefore, the acceleration will be zero at 1 s.

$s(1) = 1 - 10$

$= -9$

< 0

$v(1) = 5 - 20$

$= -15$

< 0

Since the signs of both s and v are the same at $t = 1$, the object is moving away from the origin at that time.

15. a. $s(t) = kt^2 + (6k^2 - 10k)t + 2k$

$v(t) = 2kt + (6k^2 - 10k)$

$a(t) = 2k + 0$

$= 2k$

Since $k \neq 0$ and $k \in R$, then $a(t) = 2k \neq 0$ and an element of the Real numbers. Therefore, the acceleration is constant.

b. For $v(t) = 0$

$2kt + 6k^2 - 10k = 0$

$2kt = 10k - 6k^2$

$t = 5 - 3k$

$k \neq 0$

$s(5 - 3k) = k(5 - 3k)^2 + (6k^2 - 10k)(5 - 3k) + 2k$

$= k(25 - 30k + 9k^2) + 30k^2 - 18k^3 - 50k + 30k^2 + 2k$

$= 25k - 30k^2 + 9k^3 + 30k^2 - 18k^3 - 50k + 30k^2 + 2k$

$= -9k^3 + 30k^2 - 23k$

Therefore, the velocity is 0 at $t = 5 - 3k$, and its position at that time is $-9k^3 + 30k^2 - 23k$.

16. If the ball starts from an initial height of 2 m, then the formulas are $s(t) = 2 + 35t - 5t^2$ and $v(t) = 35 - 10t$. The height is greatest at the instant the upward velocity is 0.

For $v(t) = 0$,

$$t = \frac{35}{10}$$

$= 3.5$ s.

At $t = 3.5$,

$s(3.5) = 2 + 35(3.5) - 5(3.5)^2$

$= 2 + 122.5 - 61.25$

$= 63.25$ m.

This is much lower than the ceiling of the SkyDome. Thus, a major league pitcher is not likely to hit the ceiling.

17. a. The acceleration is continuous at $t = 0$ if $\lim\limits_{t \to 0} a(t) = a(0)$.

For $t \geq 0$,

$$s(t) = \frac{t^3}{t^2 + 1}$$

and $v(t) = \dfrac{3t^2(t^2 + 1) - 2t(t^3)}{(t^2 + 1)^2}$

$$= \frac{t^4 + 3t^2}{(t^2 + 1)^2}$$

and $a(t) = \dfrac{(4t^3 + 6t)(t^2 + 1)^2 - 2(t^2 + 1)(2t)(t^4 + 3t^2)}{(t^2 + 1)^2}$

$$= \frac{(4t^3 + 6t)(t^2 + 1) - 4t(t^4 + 3t^2)}{(t^2 + 1)^3}$$

$$= \frac{4t^5 + 6t^3 + 4t^3 + 6t - 4t^5 - 12t^3}{(t^2 + 1)^3}$$

$$= \frac{-2t^3 + 6t}{(t^2 + 1)^3}$$

Therefore, $a(t) = \begin{cases} 0, \ t < 0 \\ \dfrac{-2t^3 + 6t}{(t^2 + 1)^3}, \ t \geq 0 \end{cases}$

and $v(t) = \begin{cases} 0, \ t < 0 \\ \dfrac{t^4 + 3t^2}{(t^2 + 1)^2}, \ t \geq 0 \end{cases}$

$\lim_{t \to 0^-} a(t) = 0, \ \lim_{t \to 0^+} a(t) = \dfrac{0}{1}$

$= 0.$

Thus, $\lim_{t \to 0} a(t) = 0.$

Also, $a(0) = \dfrac{0}{1}$

$= 0.$

Therefore, $\lim_{t \to 0} a(t) = a(0).$

Thus, the acceleration is continuous at $t = 0$.

b. $\lim_{t \to +\infty} v(t) = \lim_{t \to +\infty} \dfrac{t^4 + 3t^2}{t^4 + 2t^2 + 1}$

$$= \lim_{t \to +\infty} \frac{1 + \frac{3}{t^2}}{1 + \frac{2}{t^2} + \frac{1}{t^4}}$$

$= 1$

$$\lim_{t \to +\infty} a(t) = \lim_{t \to \infty} \frac{\frac{-2}{t^3} + \frac{6}{t^4}}{1 + \frac{3}{t^2} + \frac{3}{t^4} + \frac{1}{t^6}}$$

$= \dfrac{0}{1}$

$= 0$

18. $v = \sqrt{b^2 + 2gs}$

$v = (b^2 + 2gs)^{\frac{1}{2}}$

$$\frac{dv}{dt} = \frac{1}{2}\left(b^2 + 2gs\right)^{-\frac{1}{2}} \bullet \left(0 + 2g\frac{ds}{dt}\right)$$

$a = \dfrac{1}{2v} \bullet 2gv$

$a = g$

Since g is a constant, a is a constant, as required.

Note: $\dfrac{ds}{dt} = v$

$\dfrac{dv}{dt} = a$

19. $F = \dfrac{m_0 a}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{3}{2}}}$

$$F = m_0 \frac{d\left(\dfrac{v}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}\right)}{dt}$$

Using the quotient rule,

$$= \frac{m_0 \frac{dv}{dt}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{2v\frac{dv}{dt}}{c^2}\right) \bullet v}{1 - \frac{v^2}{c^2}}$$

Since $\dfrac{dv}{dt} = a$,

$$= \frac{m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left[a\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2 a}{c^2}\right]}{1 - \frac{v^2}{c^2}}$$

$$= \frac{m_0\left[\frac{ac^2 - av^2}{c^2} + \frac{v^2 a}{c^2}\right]}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$$= \frac{m_0 ac^2}{c^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$= \dfrac{m_0 a}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$, as required.

Exercise 5.3

2. $T(x) = \dfrac{200}{1 + x^2}$

a. $\dfrac{dx}{dt} = 2$ m/s

Find $\dfrac{dT(x)}{dt}$ when $x = 5$ m:

$T(x) = \dfrac{200}{1 + x^2}$

$= 200(1 + x^2)^{-1}$

$\dfrac{dT(x)}{dt} = -200(1 + x^2)^{-2}\, 2x \dfrac{dx}{dt}$

$= \dfrac{-400x}{(1 + x^2)^2} \bullet \dfrac{dx}{dt}.$

At a specific time, when $x = 5$,

$$\frac{dT(5)}{dt} = \frac{-400(5)}{(26)^2} \qquad (2)$$

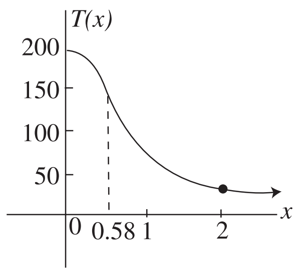
$$= \frac{-4000}{676}$$

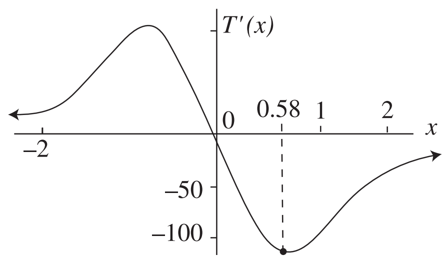
$$= \frac{-1000}{169}$$

$$\frac{dT(5)}{dt} \doteq -5.9.$$

Therefore, the temperature is decreasing at a rate of 5.9°C per s.

b.





c. Solve $T''(x) = 0$.

$$T'(x) = \frac{-400x}{(1 + x^2)^2}$$

$$T''(x) = \frac{-400(1 + x^2)^2 - 2(1 + x^2)(2x)(-400x)}{(1 + x^2)^4}$$

Let $T''(x) = 0$,

$-400(1 + x^2)^2 + 1600x^2(1 + x^2) = 0.$

Divide,

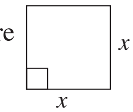
$400(1 + x^2) - (1 + x^2) + 4x^2 = 0$

$3x^2 = 1$

$$x^2 = \frac{1}{\sqrt{3}}$$

$$x = \frac{1}{\sqrt{3}}$$

$x > 0$ or $x \doteq 0.58.$

3. Given square  $\frac{dx}{dt} = 5$ cm/s.

Find $\frac{dA}{dt}$ when $x = 10$ cm.

Solution

Let the side of a square be x cm.

$A = x^2$

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

At a specific time, $x = 10$ cm.

$$\frac{dA}{dt} = 2(10)(5)$$

$$= 100$$

Therefore, the area is increasing at 100 cm²/s when a side is 10 cm.

$P = 4x$

$$\frac{dP}{dt} = 4\frac{dx}{dt}$$

At any time, $\frac{dx}{dt} = 5.$

$$\therefore \frac{dP}{dt} = 20.$$

Therefore, the perimeter is increasing at

20 cm/s.

4. Given cube with sides x cm,

$$\frac{dx}{dt} = 5 \text{ cm/s}.$$

a. Find $\frac{dv}{dt}$ when $x = 5$ cm:

$V = x^3$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

At a specific time, $x = 5$ cm.

$$\frac{dV}{dt} = 3(5)^2(4)$$

$$= 300$$

Therefore, the volume is increasing at 300 cm³/s.

b. Find $\frac{dS}{dt}$ when $x = 7$ cm.

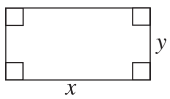
$S = 6x^2$

$\frac{dS}{dt} = 12x\frac{dx}{dt}$

At a specific time, $x = 7$ cm,

$$\begin{aligned}\frac{dS}{dt} &= 12(7)(4)\\ &= 336.\end{aligned}$$

Therefore, the surface area is increasing at a rate of 336 cm^2/s.

5. Given rectangle 

$\frac{dx}{dt} = 2$ cm/s

$\frac{dy}{dt} = -3$ cm/s

Find $\frac{dA}{dt}$ when $x = 20$ cm and $y = 50$ cm.

Solution

$A = xy$

$\frac{dA}{dt} = \frac{dx}{dt}y + \frac{dy}{dt}x$

At a specific time, $x = 20$, $y = 50$,

$$\begin{aligned}\frac{dA}{dt} &= (2)(50) + (-3)(20)\\ &= 100 - 50\\ &= 40.\end{aligned}$$

Therefore, the area is increasing at a rate of 40 cm^2/s.

6. Given circle with radius r,

$\frac{dA}{dt} = -5\ \text{m}^2\text{/s}.$

a. Find $\frac{dr}{dt}$ when $r = 3$ m.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

When $r = 3$,

$-5 = 2\pi(3)\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-5}{6\pi}.$

Therefore, the radius is decreasing at a rate of $\frac{5}{6\pi}$ m/s when $r = 3$ m.

b. Find $\frac{dD}{dt}$ when $r = 3$.

$$\begin{aligned}\frac{dD}{dt} &= 2\frac{dr}{dt}\\ &= 2\left(\frac{-5}{6\pi}\right)\\ &= \frac{-5}{3\pi}\end{aligned}$$

Therefore, the diameter is decreasing at a rate of $\frac{5}{3\pi}$ m/s.

7. Given circle with radius r,

$\frac{dA}{dt} = 6\ \text{km}^2\text{/h}$

Find $\frac{dr}{dt}$ when $A = 9\pi\ \text{km}^2$.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

When $A = 9\pi$,

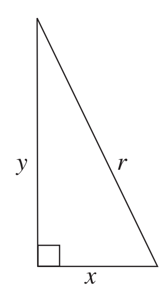
$$\begin{aligned}9\pi &= \pi r^2\\ r^2 &= 9\\ r &= 3\\ r &> 0.\end{aligned}$$

When $r = 3$,

$$\begin{aligned}\frac{dA}{dt} &= 6\\ 6 &= 2\pi(3)\frac{dr}{dt}\\ \frac{dr}{dt} &= \frac{1}{\pi}.\end{aligned}$$

Therefore, the radius is increasing at a rate of $\frac{1}{\pi}$ km/h.

8.



Let x represent the distance from the wall and y the height of the ladder on the wall.

$x^2 + y^2 = r^2$

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2r\dfrac{dr}{dt}$

$x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = r\dfrac{dr}{dt}$

When $r = 5$, $y = 3$,

$x^2 = 25 - 9$

$\quad = 16$

$x = 4$

$x = 4, y = 3, r = 5$

$\dfrac{dx}{dt} = \dfrac{1}{3}, \dfrac{dr}{dt} = 0.$

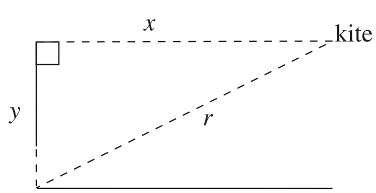
Substituting,

$4\left(\dfrac{1}{3}\right) + 3\left(\dfrac{dy}{dt}\right) = 5(0)$

$\dfrac{dy}{dt} = -\dfrac{4}{9}.$

Therefore, the top of the ladder is sliding down at 4 m/s.

9.



Let the variables represent the distances as shown on the diagram.

$x^2 + y^2 = r^2$

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2r\dfrac{dr}{dt}$

$x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = r\dfrac{dr}{dt}$

$x = 30, y = 40$

$r^2 = 30^2 + 40^2$

$r = 50$

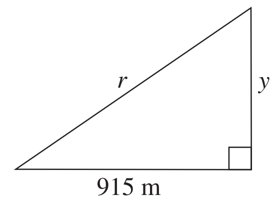
$\dfrac{dr}{dt} = ?, \dfrac{dx}{dt} = 10, \dfrac{dy}{dt} = 0$

$30(10) + 40(0) = 50\left(\dfrac{dr}{dt}\right)$

$\dfrac{dr}{dt} = 8$

Therefore, she must let out the line at a rate of 8 m/min.

10.



Label diagram as shown.

$r^2 = y^2 + 915^2$

$2r\dfrac{dr}{dt} = 2y\dfrac{dy}{dt}$

$r\dfrac{dr}{dt} = y\dfrac{dy}{dt}$

When $y = 1220$, $\dfrac{dy}{dt} = 268$ m/s.

$r = \sqrt{1220^2 + 915^2}$

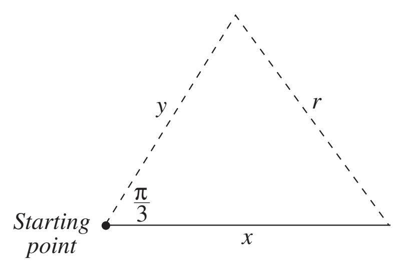
$\quad = 1525$

$\therefore\ 1525\left(\dfrac{dr}{dt}\right) = 1220 \times 268$

$\dfrac{dr}{dt} = 214$ m/s

Therefore, the camera–to–rocket distance is changing at 214 m/s.

11.



$\dfrac{dx}{dt} = 15$ km/h

$\dfrac{dy}{dt} = 20$ km/h

Find $\dfrac{dr}{dt}$ when $t = 2$ h.

Solution

Let x represent the distance cyclist 1 is from the starting point, $x \geq 0$. Let y represent the distance cyclist 2 is from the starting point, $y \geq 0$ and let r be the distance the cyclists are apart. Using the cosine law,

$r^2 = x^2 + y^2 - 2xy\cos\dfrac{\pi}{3}$

$\quad = x^2 + y^2 - 2xy\left(\dfrac{1}{2}\right)$

$r^2 = x^2 + y^2 - xy$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \left[\frac{dx}{dt}y + \frac{dy}{dt}x\right]$$

At $t = 2$ h, $x = 30$ km, $y = 40$ km

and $r^2 = 30^2 + 40^2 - 2(30)(40)\cos\frac{\pi}{3}$

$= 2500 - 2(1200)\left(\frac{1}{2}\right)$

$= 1300$

$r = 10\sqrt{13}$, $r > 0$.

$$\therefore\ 2(10\sqrt{13})\frac{dr}{dt} = 2(30)(15) + 2(40)(20) - [15(40) + 20(30)]$$

$$20\sqrt{13}\frac{dr}{dt} = 900 + 1600 - [600 - 600]$$

$$= 1300$$

$$\frac{dr}{dt} = \frac{130}{2\sqrt{13}}$$

$$= \frac{65}{\sqrt{13}}$$

$$= \frac{65\sqrt{13}}{13}$$

$$= 5\sqrt{13}$$

Therefore, the distance between the cyclists is increasing at a rate of $5\sqrt{13}$ km/h after 2 h.

12. Given sphere $v = \frac{4}{3}\pi r^3$

$\frac{dv}{dt} = 8$ cm^3/s.

a. Find $\frac{dr}{dt}$ when $r = 12$ cm.

$$v = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$$

At a specific time, when $r = 12$ cm:

$$8 = 4\pi(12)^2\frac{dr}{dt}$$

$$8 = 4\pi(144)\frac{dr}{dt}$$

$$\frac{1}{72\pi} = \frac{dr}{dt}.$$

Therefore, the radius is increasing at a rate of $\frac{1}{72\pi}$ cm/s.

b. Find $\frac{dr}{dt}$ when $v = 1435$ cm^3.

Solution

$$v = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}.$$

At a specific time, when $v = 1435$ cm^3:

$$v = 1435$$

$$\frac{4}{3}\pi^3 = 1435$$

$$r^3 \doteq 342.581015$$

$$\doteq 6.9971486$$

$$= 7$$

$$8 \doteq 4\pi(7)^2\frac{dr}{dt}$$

$$8 = 196\pi\frac{dr}{dt}$$

$$\frac{2}{49\pi} = \frac{dr}{dt}$$

$$0.01 = \frac{dr}{dt}.$$

Therefore, the radius is increasing at approximately $\frac{2}{49\pi}$ cm/s (or 0.01 cm/s).

c. Find $\frac{dr}{dt}$ when $t = 33.5$ s.

When $t = 33.5$, $v = 8 \times 33.5$ cm^3:

$$\frac{4}{3}\pi r^3 = 268$$

$$r^3 \doteq 63.98028712$$

$$r \doteq 3.999589273$$

$$\doteq 4.$$

Solution

$$v = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$$

At $t = 33.5$ s,

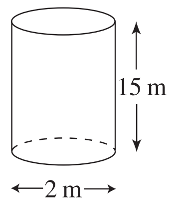
$$8 \doteq 4\pi(4)^2\frac{dr}{dt}$$

$$8 = 64\pi\frac{dr}{dt}$$

$$\frac{1}{8\pi} = \frac{dr}{dt}.$$

Therefore, the radius is increasing at a rate of $\frac{1}{8\pi}$ cm/s (or 0.04 cm/s).

13. Given cylinder



$v = \pi r^2 h$

$\frac{dv}{dt} = 500$ L/min

$= 500\ 000$ cm/min

a. Find $\frac{dy}{dt}$.

$v = \pi r^2 h$

Since the diameter is constant at 2 m, the radius is also constant at 1 m = 100 cm.

$\therefore\ v = 10\ 000\ \pi h$

$\frac{dv}{dt} = 10\ 000\pi \frac{dh}{dt}$

$500\ 000 = 10\ 000\pi \frac{dh}{dt}$

$\frac{50}{\pi} = \frac{dh}{dt}$

Therefore, the fluid level is rising at a rate of $\frac{50}{\pi}$ cm/min.

b. Find t, the time to fill the cylinder.

$V = \pi r^2 h$

$V = \pi(100)^2(1500)$ cm^3

$V = 15\ 000\ 000\pi$ cm^3

Since $\frac{dv}{dt} = 500\ 000$ cm^3/min,

it takes $\frac{15\ 000\ 000\pi}{500\ 000}$ min,

$= 30\pi$ min to fill

$\doteq 94.25$ min.

Therefore, it will take 94.25 min, or just over 1.5 h to fill the cylindrical tank.

14. There are many possible problems.

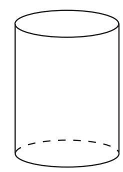
Samples:

i) The diameter of a right-circular cone is expanding at a rate of 4 cm/min. Its height remains constant at 10 cm. Find its radius when the volume is increasing at a rate of 80π cm^3/min.

ii) Water is being poured into a right-circular tank at the rate of 12π m^3/min. Its height is 4 m and its radius is 1 m. At what rate is the water level rising?

iii) The volume of a right-circular cone is expanding because its radius is increasing at 12 cm/min and its height is increasing at 6 cm/min. Find the rate at which its volume is changing when its radius is 20 cm and its height is 40 cm.

15. Given cylinder



$d = 1$ m

$h = 15$ m

$\frac{dr}{dt} = 0.003$ m/annum

$\frac{dh}{dt} = 0.4$ m/annum

Find $\frac{dv}{dt}$ at the instant $D = 1$

$v = \pi r^2 h$

$\frac{dv}{dt} = \left(2\pi r \frac{dr}{dt}\right)(h) + \left(\frac{dh}{dt}\right)(\pi r^2).$

At a specific time, when $D = 1$; i.e., $r = 0.5$,

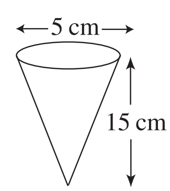
$\frac{dv}{dt} = 2\pi(0.5)(0.003)(15) + 0.4\pi(0.5)^2$

$= 0.045\pi + 0.1\pi$

$= 0.145\pi$

Therefore, the volume of the trunk is increasing at a rate of 0.145π m^3/annum.

16. Given cone



$r = 5$ cm

$h = 15$ cm

$\frac{dv}{dt} = 2$ cm^3/min

Find $\frac{dh}{dt}$ when $h = 3$ cm,

$v = \frac{1}{3}\pi r^2 h$.

Using similar triangles, $\frac{r}{h} = \frac{5}{15} = \frac{1}{3}$

$$\therefore r = \frac{h}{3}.$$

Substituting into $v = \frac{1}{3}\pi r^2 h$,

$$v = \frac{1}{3}\pi\left(\frac{h^2}{9}\right)h$$

$$= \frac{1}{27}\pi h^3$$

$$\frac{dv}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

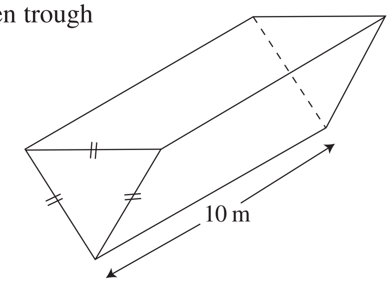
At a specific time, when $h = 3$ cm,

$$-2 = \frac{1}{9}\pi(3)^2 \frac{dh}{dt}$$

$$-\frac{2}{\pi} = \frac{dh}{dt}.$$

Therefore, the water level is being lowered at a rate of $\frac{2}{\pi}$ cm/min when height is 3 cm.

17. Given trough

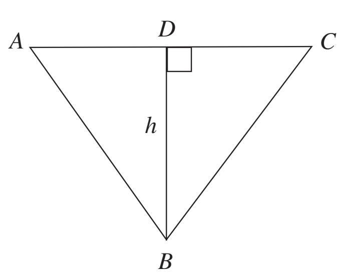


Find a formula for the volume.

v = area of a cross section × length

= area of an equilateral triangle × 10

Let h be the height of any cross section.



Since $\angle C = 60°$, $\angle B = 30°$ and ΔDBC is a special triangle similar to the 1, $\sqrt{3}$, 2 triangle.

Since $DB = h$, then $DC = \frac{h}{\sqrt{3}}$ from similar triangles.

Therefore, $AC = \frac{2h}{\sqrt{3}}$

$$v = \frac{1}{2}AC \times DB \times 10$$

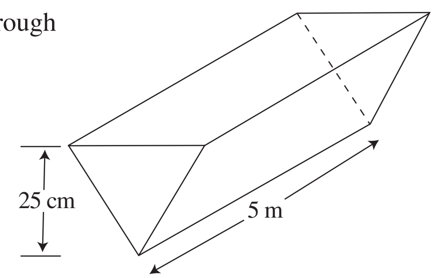
$$= \frac{1}{2} \times \frac{2h}{\sqrt{3}} \times h \times 10$$

$$= \frac{h^2}{\sqrt{3}} \times 10$$

$$= \frac{10h^2}{\sqrt{3}}.$$

Therefore, the volume of the trough of height h is given by $v = \frac{10h^2}{\sqrt{3}}$.

18. Given trough



$$\frac{dv}{dt} = 0.25 \frac{\text{m}^3}{\text{min}}$$

Find $\frac{dh}{dt}$ when $h = 10$ cm

$$= 0.1 \text{ m}.$$

Since the cross section is equilateral, the $v = \frac{h^2}{\sqrt{3}} \times \ell$.

$$v = \frac{h^2}{\sqrt{3}} \times 5.$$

$$\frac{dv}{dt} = \frac{10}{\sqrt{3}} h \frac{dh}{dt}$$

At a specific, time when $h = 0.1 = \frac{1}{10}$,

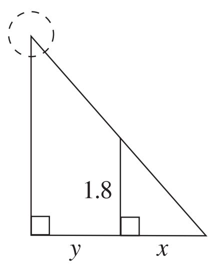
$$0.25 = \frac{10}{\sqrt{3}} \frac{1}{10} \frac{dh}{dt}$$

$$0.25\sqrt{3} = \frac{dh}{dt}$$

$$\frac{\sqrt{3}}{4} = \frac{dh}{dt}$$

Therefore, the water level is rising at a rate of $\frac{\sqrt{3}}{4}$ m/min.

19. Given

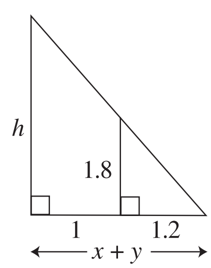


$\frac{dy}{dt} = 120$ m/min

Find $\frac{dx}{dt}$ when $t = 5$ s.

Solution

Let x represent the length of the shadow. Let y represent the distance the man is from the base of the lamppost. Let h represent the height of the lamppost. At a specific instant, we have



Using similar triangles,

$$\frac{x+y}{h} = \frac{1.2}{1.8}$$

$$\frac{2.2}{h} = \frac{2}{3}$$

$$2h = 6.6$$

$$h = 3.3$$

Therefore, the lamppost is 3.3 m high.

At any time,

$$\frac{x+y}{x} = \frac{3.3}{1.8}$$

$$\frac{x+y}{x} = \frac{11}{6}$$

$$6x + 6y = 11x$$

$$6y = 5x$$

$$6\frac{dy}{dt} = 5\frac{dx}{dt}.$$

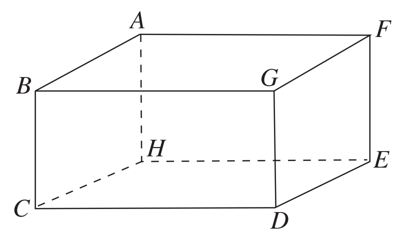
At a specific time, when $t = 5$ seconds $\frac{dy}{dt} = 120$ m/min,

$$6 \times 120 \quad = 5\frac{dx}{dt}$$

$$\frac{dx}{dt} = 144.$$

Therefore, the man's shadow is lengthening at a rate of 144 m/min after 5 s.

20. This question is similar to finding the rate of change of the length of the diagonal of a rectangular prism.



$$20 \text{ m} = \frac{20}{1000} \text{ km}$$

$$= \frac{1}{50} \text{ km}$$

Find $\frac{d(GH)}{dt}$ *at* $t = 10$ s,

$$= \frac{1}{360} \text{ h}.$$

Let BG be the path of the train and CH be the path of the boat:

$\therefore \frac{d(BG)}{dt} = 60$ km/h and $\frac{d(CH)}{dt} = 20$ km/h.

At $t = \frac{1}{360}$h, $BG = 60 \times \frac{1}{360}$

$$= \frac{1}{6} \text{ km}$$

and $CH = 20 \times \frac{1}{360}$

$$= \frac{1}{18} \text{ km}.$$

Using the Pythagorean Theorem,

$GH^2 = HD^2 + DG^2$

and $HD^2 = CD^2 + CH^2$

$\therefore GH^2 = CD^2 + CH^2 + DG^2$

Since $BG = CD$ and $FE = GD = \frac{1}{50}$, it follows that

$GH^2 = BG^2 + CH^2 + \frac{1}{2500}$.

$$2(GH)\frac{d(GH)}{dt} = 2(BG)\frac{d(BG)}{dt} + 2(CH)\frac{d(CH)}{dt}$$

At $t = 10$ s,

$$GH\frac{d(GH)}{dt} = \frac{1}{6}(60) + \frac{1}{18}(20)$$

$$\frac{\sqrt{6331}}{450}\frac{d(GH)}{dt} = \frac{100}{9}$$

$$\frac{d(GH)}{dt} = \frac{45\,000}{9\sqrt{6331}}$$

$$\doteq 62.8.$$

$$\text{And } GH^2 = \left(\frac{1}{6}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{1}{50}\right)^2$$

$$= \frac{1}{36} + \frac{1}{324} + \frac{1}{2500}$$

$$= \frac{911\,664}{29\,160\,000} \div 8$$

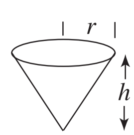
$$GH^2 = \frac{113\,958}{364\,500} \div 18$$

$$= \frac{6331}{202\,500}$$

$$GH = \frac{\sqrt{6331}}{450} = \frac{\sqrt{13 \times 487}}{450}$$

Therefore, they are separating at a rate of approximately 62.8 km/h.

21. Given cone



$r = h$

$$\frac{dv}{dt} = 200 - 20$$

$$= 180 \text{ cm}^3/\text{s}$$

Find $\frac{dh}{dt}$ when $h = 15$ cm.

Solution

$$v = \frac{1}{3}\pi r^2 h \text{ and } r = h$$

$$\therefore v = \frac{1}{3}\pi h^3.$$

$$\frac{dv}{dt} = \pi h^2 \frac{dh}{dt}$$

At a specific time, $h = 15$ cm.

$$180 = \pi(15)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{5\pi}$$

Therefore, the height of the water in the funnel is increasing at a rate of $\frac{4}{5\pi}$ cm/s.

Part 2

$$\frac{dv}{dt} = 200 \text{ cm}^3/\text{s}$$

Find $\frac{dh}{dt}$ when $h = 25$ cm.

Solution

$$\frac{dv}{dt} = \pi h^2 \frac{dh}{dt}.$$

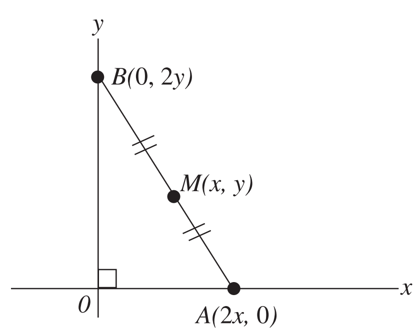
At the time when the funnel is clogged, $h = 25$ cm:

$$200 = \pi(25)^2 \frac{dh}{dt}.$$

$$\frac{dh}{dt} = \frac{8}{25\pi}.$$

Therefore, the height is increasing at $\frac{8}{25\pi}$ cm/s.

22.



Let the midpoint of the ladder be (x, y). From similar triangles, it can be shown that the top of the ladder and base of the ladder would have points $B(0, 2y)$ and $A(2x, 0)$ respectively. Since the ladder has length ℓ,

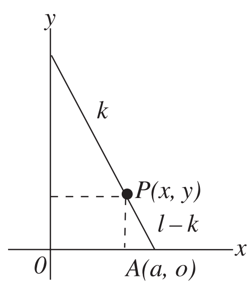
$(2x)^2 + (2y)^2 = \ell^2$

$4x^2 + 4y^2 = \ell^2$

$x^2 + y^2 = \dfrac{\ell^2}{4}$

$= \left(\dfrac{\ell}{2}\right)^2$ is the required equation.

Therefore, the equation of the path followed by the midpoint of the ladder represents a quarter circle with centre (0, 0) and radius $\frac{\ell}{2}$, with $x, y \geq 0$.



Let $P(x, y)$ be a general point on the ladder a distance k from the top of the ladder. Let $A(a, o)$ be the point of contact of the ladder with the ground.

From similar triangles, $\frac{a}{\ell} = \frac{x}{k}$ or $a = \frac{x\ell}{k}$.

Using the Pythagorean Theorem: $y^2 + (a - x)^2 = (\ell - k)^2$, and substituting $a = \frac{x\ell}{k}$,

$$y^2 + \left(\frac{x\ell}{k} - x\right)^2 = (\ell - k)^2$$

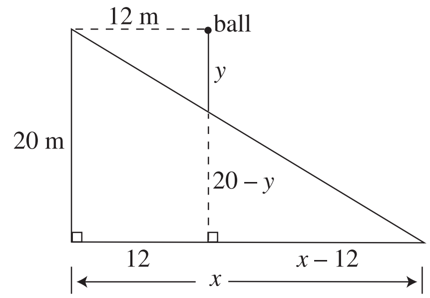
$$y^2 + x^2\left(\frac{\ell - k}{k}\right)^2 = (\ell - k)^2$$

$$\frac{(\ell - k)^2}{k^2}x^2 + y^2 = (\ell - k)^2$$

$$\frac{x^2}{k^2} + \frac{y^2}{(\ell - k)^2} = 1 \text{ is the required equation.}$$

Therefore, the equation is the first quadrant portion of an ellipse.

23.



Let x represent the distance the tip of the ball's shadow is from the base of the lamppost.

Let $\frac{dx}{dt}$ represent the rate at which the shadow is moving along the ground. Let y represent the distance the ball has fallen.

From similar triangles,

$$\frac{20 - y}{x - 12} = \frac{20}{x}$$

$$20x - xy = 20x - 240$$

$$xy = 240$$

$$\frac{dx}{dt}y + \frac{dy}{dt}x = 0.$$

At a specific time,

$$\frac{dx}{dt}(5) + (10)(48) = 0$$

$$\frac{dx}{dt} = -\frac{480}{5}$$

$$= -96.$$

Therefore, the shadow is moving at a rate of 96 m/s.
At any time, t, the height of the ball is $h = 20 - 5t^2$.
When $t = 1$, $h = 20 - 5$,
$= 15$
$\therefore y = 5$.
Also $v = -10t$ and since y increases, $\frac{dy}{dt} = 10$ when $t = 1$.

Section 5.4

Investigation

1. **a.** $f(x) = -x^2 + 6x - 3,\ 0 \le x \le 5$
$= -(x^2 - 6x + 9 - 9) - 3$
$= -(x - 3)^2 + 6$
maximum of 6 when $x = 3$

b. $f(x) = -x^2 - 2x + 11,\ -3 \le x \le 4$
$= -(x^2 + 2x + 1 - 1) + 11$
$= -(x + 1)^2 + 12$
maximum of 12 when $x = -1$

c. $f(x) = 4x^2 - 12x + 7,\ -1 \le x \le 4$
$$= 4\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) + 7$$
$$= 4\left(x - \frac{3}{2}\right)^2 - 2$$
minimum value of -2 when $x = \frac{3}{2}$

2. **a.** $f'(x) = -2x + 6 = 0$
$x = 3,\ c = 3$

b. $f'(x) = -2x - 2$
$x = -1,\ c = -1$

c. $f'(x) = 8x - 12 = 0$
$x = \frac{3}{2},\ c = \frac{3}{2}$

3. The values are the same.

4. **a.** $f(x) = x^3 - 3x^2 - 8x + 10,\ -2 \le x \le 4$
max at $x \doteq -0.91$, min at $x \doteq 2.91$

b. $f(x) = x^3 - 12x + 5,\ -3 \le x \le 3$
max at $x \doteq -1.98$, min at $x \doteq 1.98$

c. $f(x) = 3x^3 - 15x^2 + 9x + 23,\ 0 \le x \le 4$
max at $x \doteq 0.34$, min at $x \doteq 2.98$

d. $f(x) = -2x^3 + 12x + 7,\ -2 \le x \le 2$
max at $x \doteq 1.41$, min at $x \doteq -1.41$

e. $f(x) = -x^3 - 2x^2 + 15x + 23,\ -4 \le x \le 3$
max at $x \doteq 1.66$, min at $x \doteq -3.03$

5. **a.** $f'(x) = 3x^2 - 6x - 8 = 0$
$$x = \frac{6 \pm \sqrt{36 + 96}}{6}$$
$$= \frac{6 \pm \sqrt{132}}{6}$$
$x \doteq 2.91$ or $x \doteq -0.91$

b. $f'(x) = 3x^2 - 12 = 0$
$x^2 - 4 = 0$
$x = \pm 2$

c. $f'(x) = 9x^2 - 30x + 9 = 0$
$3x^2 - 10x + 3 = 0$
$(3x - 1)(x - 3) = 0$
$x = \frac{1}{3}$ or $x = 3$

d. $f120(x) = -6x^2 + 12 \qquad = 0$
$x^2 - 2 = 0$
$x = \pm\sqrt{2}$
$x = 1.41$ or $x = -1.41$

e. $f'(x) = -3x^2 - 4x + 15 = 0$
$3x^2 + 4x - 15 = 0$
$(3x - 5)(x + 3) = 0$
$x = \frac{5}{3}$ or $x = -3$

6. The values are the same.

7. Set first derivative to zero.

8. **a.** $f(x) = -x^2 + 6x - 3,\ 4 \le x \le 8$
max at $x = 4$, value 5, min at $x = 8$, $y = -19$

b. $f(x) = 4x^2 - 12x + 7,\ 2 \le x \le 6$
max at $x = 6$, value -1, min at $x = 2$, $y = 79$

c. $f(x) = x^3 - 3x^2 - 9x + 10,\ -2 \le x \le 6$
max at $x = -2$, $y = 40$, min at $x = 6$, $y = -800$

d. $f(x) = x^3 - 12x + 5,\ 0 \le x \le 5$
max at $x = 5$, $y = -11$, min at $x = 2$, $y = 70$

e. $f(x) = x^3 - 5x^2 + 3x + 7,\ -2 \le x \le 5$
max at $x = 5$, $y = 20$, min at $x = -2$, $y = -29$

9. End points of the interval.

Exercise 5.4

3. **a.** $f(x) = x^2 - 4x + 3, 0 \le x \le 3$

$f'(x) = 2x - 4$

Let $2x - 4 = 0$ for max or min

$x = 2$

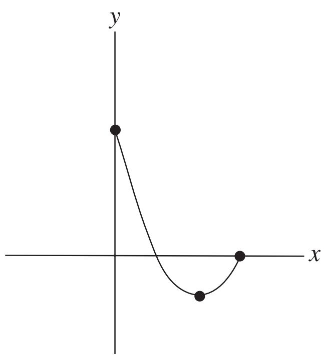
$f(0) = 3$

$f(2) = 4 - 8 + 3 = -1$

$f(3) = 9 - 12 + 3 = 0$

max is 3 at $x = 0$

min is -1 at $x = 2$



c. $f(x) = x^3 - 3x^2, -1 \le x \le 3$

$f'(x) = 3x^2 - 6x$

Let $f'(x) = 0$ for max or min

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0$ or $x = 2$

$f(-1) = -1 - 3$

$= -4$

$f(0) = 0$

$f(2) = 8 - 12$

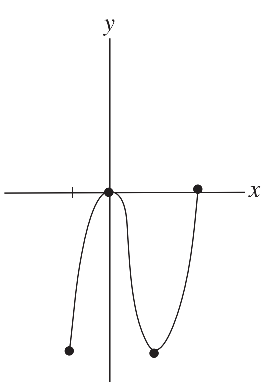
$= -4$

$f(3) = 27 - 27$

$= 0$

min is -4 at $x = -1, 2$

max is 0 at $x = 0, 3$



e. $f(x) = 2x^3 - 3x^2 - 12x + 1, -2 \le x \le 0$

$f'(x) = 6x^2 - 6x - 12$

Let $f'(x) = 0$ for max or min

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

$f(-2) = -16 - 12 + 24 + 1$

$= -3$

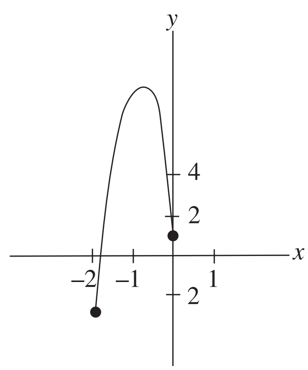
$f(-1) = 8$

$f(0) = 1$

$f(2) =$ not in region

max of 8 at $x = -1$

min of -3 at $x = -2$



4. **b.** $f(x) = 4\sqrt{x} - x, 2 \le x \le 9$

$f'(x) = 2x^{-\frac{1}{2}} - 1$

Let $f'(x) = 0$ for max or min

$\frac{2}{\sqrt{x}} - 1 = 0$

$x = \sqrt{2}$

$x = 4$

$f(2) = 4\sqrt{2} - 2 \doteq 3.6$

$f(4) = 4\sqrt{4} - 4 = 4$

$f(9) = 4\sqrt{9} - 9 = 3$

min value of 3 when $x = 9$

max value of 4 when $x = 4$

c. $f(x) = \dfrac{1}{x^2 - 2x + 2}, 0 \leq x \leq 2$

$f'(x) = -(x^2 - 2x + 2)^{-2}(2x - 2)$

$= -\dfrac{2x - 2}{(x^2 - 2x + 2)^2}$

Let $f'(x) = 0$ for max or min.

$-\dfrac{2x - 2}{(x^2 - 2x + 2)} = 0$

$\therefore 2x - 2 = 0$

$x = 1$

$f(0) = \dfrac{1}{2}, f(1) = 1, f(2) = \dfrac{1}{2}$

max value of 1 when $x = 1$

min value of $\dfrac{1}{2}$ when $x = 0, 2$

e. $f(x) = \dfrac{4x}{x^2 + 1}, -2 \leq x \leq 4$

$f'(x) = \dfrac{4(x^2 + 1) - 2x(4x)}{(x^2 + 1)^2}$

$= \dfrac{-4x^2 + 4}{x^2 + 1}$

Let $f'(x) = 0$ for max or min:

$-4x^2 + 4 = 0$

$x^2 = 1$

$x = \pm 1$

$f(-2) = \dfrac{-8}{5}$

$f(-1) = \dfrac{-4}{2}$

$= -2$

$f(1) = \dfrac{4}{2}$

$= 2$

$f(4) = \dfrac{16}{17}$

max value of 2 when $x = 1$

min value of -2 when $x = -1$

5. **a.** $v(t) = \dfrac{4t^2}{4 + t^3}, t \geq 0$

Interval $1 \leq t \leq 4$

$v(1) = \dfrac{4}{5}\ v(4)$

$= \dfrac{16}{17}$

$v'(t) = \dfrac{(4 + t^3)(8t) - 4t^2(3t^2)}{(4 + t^3)^2} = 0$

$32t + 8t^4 - 12t^4 = 0$

$-4t(t^3 - 8) = 0$

$t = 0, t = 2$

$v(2) = \dfrac{16}{12} = \dfrac{4}{3}$

max velocity is $\dfrac{4}{3}$ m/s

min velocity is $\dfrac{4}{5}$ m/s

7. **a.** $E(v) = \dfrac{1600v}{v^2 + 6400}\ 0 \leq v \leq 100$

$E'(v) = \dfrac{1600(v^2 + 6400) - 1600v(2v)}{(v^2 + 6400)^2}$

Let $E'(N) = 0$ for max or min

$\therefore 1600v^2 + 6400 \times 1600 - 3200v^2 = 0$

$1600v^2 = 6400 \times 1600$

$v = \pm 80$

$E(0) = 0$

$E(80) = 10$

$E(100) = 9.756$

The legal speed limit that maximizes fuel efficiency is 80 km/h.

8. $C(t) = \dfrac{0.1t}{(t + 3)^2}, 1 \leq t \leq 6$

$C'(t) = \dfrac{0(t + 3)^2 - 0.2t(t + 3)}{(t + 3)^4} = 0$

$(t + 3)(0.1t + 0.3 - 0.2t) = 0$

$t = 3$

$C(1) \doteq 0.00625$

$C(3) = 0.0083, C(6) \doteq 0.0074$

The min concentration is at $t = 1$ and the max concentration is at $t = 3$.

9. $P(t) = 2t + \dfrac{1}{162t + 1}, 0 \le t \le 1$

$$P'(t) = 2(162t + 1)^{-2}(162) = 0$$

$$\frac{162}{(162t + 1)^2} = 2$$

$$81 = 162^2 + t^2 + 324t + 1$$

$$162^2t^2 + 324t - 80 = 0$$

$$81^2t^2 + 81t - 20 = 0$$

$$(81t + 5)(81t - 4) = 0$$

$$t > 0 \therefore t = \frac{4}{81}$$

$$= 0.05$$

$P(0) = 1$
$P(0.05) = 0.21$
$P(1) = 2.01$

Pollution is at its lowest level in 0.05 years or approximately 18 days.

10. $r(x) = \dfrac{1}{400}\left(\dfrac{4900}{x} + x\right)$

$$r'(x)\ \frac{1}{400}\left(\frac{-4900}{x^2} + 1\right) = 0$$

Let $r'(x) = 0$

$x^2 = 4900,$
$x = 70, x > 0$
$r(30) = 0.4833$
$r(70) = 0.35$
$r(120) = 0.402$

A speed of 70 km/h uses fuel at a rate of 0.35 L/km.
Cost of trip is $0.35 \times 200 \times 0.45 = \$31.50$.

11. $C(x) = 3000 + 9x + 0.05x^2, 1 \le x \le 300$

Unit cost $u(x) = \dfrac{C(x)}{x}$

$$= \frac{3000 + 9x + 0.05x^2}{x}$$

$$= \frac{3000}{x} + 9 + 0.05x$$

$$U'(x) = \frac{-3000}{x^2} + 0.05$$

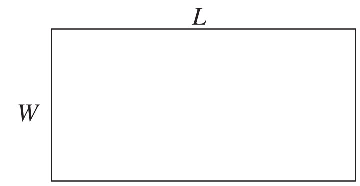
For max or min, let $U'(x) = 0$:

$0.05x^2 = 3000$
$x^2 = 60\ 000$
$x \doteq 244.9$
$U(1) = 3009.05$
$U(244) = 33.4950$
$U(245) = 33.4948$
$U(300) = 34.$

Production level of 245 units will minimize the unit cost to \$33.49.

Exercise 5.5

1.



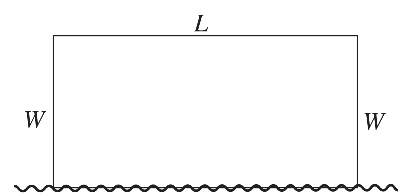
Let the length be L cm and the width be W cm.

$2(L + W) = 100$
$L + W = 50$
$L = 50 - W$
$A = L \bullet W$
$= (50 - W)(W)$
$A(W) = -W^2 + 50W$ for $0 \le W \le 50$
$A'(W) = -2W + 50$

Let $A'(W) = 0$:
$\therefore -2W + 50 = 0$
$W = 25$
$A(0) = 0$
$A(25) = 25 \times 25$
$= 625$
$A(50) = 0.$

The largest area is 625 cm^2 and occurs when $W = 25$ cm and $L = 25$ cm.

3.



Let the length of L m and the width W m.

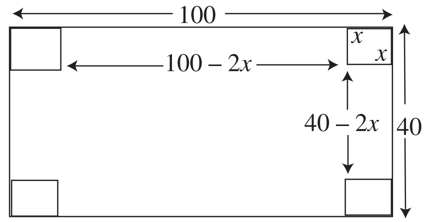
$2W + L = 600$
$L = 600 - 2W$
$A = L \bullet W$
$= W(600 - 2W)$
$A(W) = -2w^2 + 600W, 0 \le W \le 300$
$A'(W) = -4w + 600$

For max or min, let $\dfrac{dA}{dW} = 0$:

$\therefore W = 50$
$A(0) = 0$
$A(150) = -2(150)^2 + 600 \times 150$
$= 45\ 000$
$A(300) = 0.$

The largest area of 45 000 m^2 occurs when $W = 150$ m and $L = 300$ m.

4. Let dimensions of cut be x cm by x cm. Therefore, the height is x cm.



Length of the box is $100 - 2x$.
Width of the box is $40 - 2x$.
$V = (100 - 2x)(40 - 2x)(x)$ for domain $0 \le x \le 20$
Using Algorithm for Extreme Value,

$$\frac{dV}{dx} = (100 - 2x)(40 - 4x) + (40x - 2x^2)(-2)$$
$$= 4000 - 480x + 8x^2 - 80x + 4x^2$$
$$= 12x^2 - 560x + 4000$$

Set $\frac{dV}{dx} = 0$

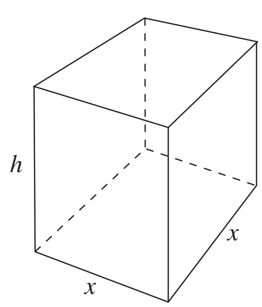
$3x^2 - 140x + 1000 = 0$

$$x = \frac{140 \pm \sqrt{7600}}{6}$$

$$x = \frac{140 \pm 128.8}{6}$$

$x = 8.8$ or $x = 37.9$
Reject $x = 37.9$ since $0 \le x \le 20$
When $x = 0$, $V = 0$
$x = 8.8$, $V = 28\,850$ cm^2
$x = 20$, $V = 0$.
Therefore, the box has a height of 8.8 cm, a length of $100 - 2 \times 8.8 = 82.4$ cm, and a width of $40 - 3 \times 8.8 = 22.4$ cm.

5.



Let the base be x by x and the height be h
$x^2h = 1000$

$$\therefore h = \frac{1000}{x^2} \qquad (1)$$

Surface area $= 2x^2 + 4xh$
$$A = 2x^2 + 4xh \qquad (2)$$
$$= 2x^2 + 4x\left(\frac{1000}{x^2}\right)$$
$$= 2x^2 + \frac{4000}{x} \text{ for domain } 0 \le x \le 10\sqrt{2}$$

Using the max min Algorithm,

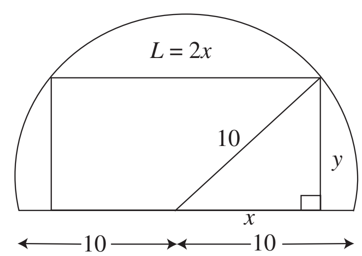
$$\frac{dA}{dx} = 4x - \frac{4000}{x^2} = 0$$

$x \ne 0$, $4x^3 = 4000$
$x^3 = 1000$
$x = 10$
$\therefore A = 200 + 400 = 600$ cm^2
Step 2: At $x \to 0$, $A \to \infty$

Step 3: At $x = 10\sqrt{10}$, $A = 2000 + \frac{4000}{10\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$
$= 2000 + 40\sqrt{10}$

Minimum area is 600 cm^2 when the base of the box is 10 cm by 10 cm and height is 10 cm.

6.



Let the length be $2x$ and the height be y. We know $x^2 + y^2 = 100$.
$\therefore y = \pm\sqrt{100 - x^2}$
Omit negative area $= 2xy$
$= 2x\sqrt{100 - x^2}$ for domain $0 \le x \le 10$
Using the max min Algorithm,

$$\frac{dA}{dx} = 2\sqrt{100 - x^2} + 2y \bullet \frac{1}{2}(100 - x^2)^{-\frac{1}{2}}(-2x).$$

Let $\frac{dA}{dx} = 0$.

$$\therefore\ 2\sqrt{100 - x^2} - \frac{2x^2}{\sqrt{100 - x^2}} = 0$$

$$\therefore\ 2(100 - x^2) - 2x^2 = 0$$

$$\therefore\ 100 = 2x^2$$

$$x^2 = 50$$

$x = 5\sqrt{2}$, $x > 0$. Thus, $y = 5\sqrt{2}$, $L = 10\sqrt{2}$

Part 2: If $x = 0$, $A = 0$

Part 3: If $x = 10$, $A = 0$

The largest area occurs when $x = 5\sqrt{2}$ and the area is

$10\sqrt{2}\sqrt{100 - 50}$

$= 10\sqrt{2}\sqrt{50}$

$= 100$ square units.

7. **a.** Let the radius be r cm and the height be h cm.

Then $\pi r^2 h = 1000$

$$h = \frac{1000}{\pi r^2}$$

Surface Area: $A = 2\pi r^2 + 2\pi rh$

$$= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{4},\ 0 \le\ r \le\ \infty$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

For max or min, let $\frac{dA}{dr} = 0$.

$$4\pi r - \frac{2000}{r^2} = 0$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \doteq 5.42$$

When $r = 0$, $A \to \infty$

$r = 5.42$ $A \doteq 660.8$

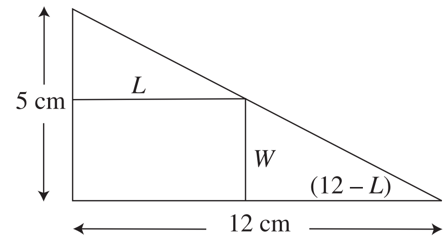
$r \to \infty$, $A \to \infty$

The minimum surface area is approximately $661\ \text{cm}^3$ when $r = 5.42$.

b. $r = 5.42$, $h = \frac{1000}{\pi(5.42)^2} \doteq 10.84$

$$\frac{h}{d} = \frac{10.84}{2 \times 5.42} = \frac{1}{1}$$

8. **a.**



Let the rectangle have length L cm on the 12 cm leg and width W cm on the 5 cm leg.

$A = LW$ (1)

By similar triangles, $\frac{12 - L}{12} = \frac{W}{5}$

$$\therefore\ 60 - 5L = 12W$$

$$L = \frac{60 - 12W}{5} \qquad (2)$$

$$A = \frac{(60 - 12W)W}{5} \text{ for domain } 0 \le\ W \le\ 5$$

Using the max min Algorithm,

$$\frac{dA}{dW} = \frac{1}{5}[60 - 24W] = 0,\ W = \frac{60}{24} = 2.5 \text{ cm.}$$

When $W = 2.5$ cm, $A = \frac{(60 - 30) \times 2.5}{5} = 15\ \text{cm}^2$.

Step 2: If $W = 0$, $A = 0$

Step 3: If $W = 5$, $A = 0$

The largest possible area is $15\ \text{cm}^2$ and occurs when $W = 2.5$ cm and $L = 6$ cm.

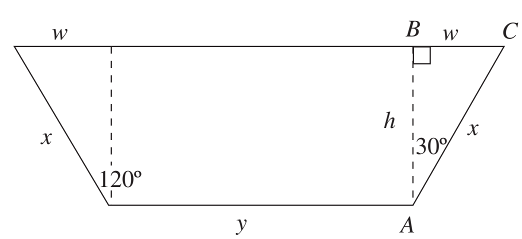
9. **a.** Let the base be y cm, each side x cm and the height h cm.

$2x + y = 60$

$y = 60 - 2x$

$$A = yh + 2 \times \frac{1}{2}(wh)$$

$$= yh + wh$$



From ΔABC

$\frac{h}{x} = \cos 30°$

$h = x \cos 30°$

$= \frac{\sqrt{3}}{2} x$

$\frac{w}{x} = \sin 30°$

$w = x \sin 30°$

$= \frac{1}{2} x$

Therefore, $A = (60 - 2x)\left(\frac{\sqrt{3}}{2} x\right) + \frac{x}{2} \times \frac{\sqrt{3}}{2}x$

$A(x) = 30\sqrt{3}x - \sqrt{3}x^2 + \frac{\sqrt{3}}{4} x^2, 0 \le x \le 30$

Apply the Algorithm for Extreme Values,

$A'(x) = 30\sqrt{3} - 2\sqrt{3}x + \frac{\sqrt{3}}{2} x$

Now, set $A'(x) = 0$

$30\sqrt{3} - 2\sqrt{3}x + \frac{\sqrt{3}}{2} x = 0.$

Divide by $\sqrt{3}$:

$30 - 2x + \frac{x}{2} = 0$

$x = 20.$

To find the largest area, substitute $x = 0$, 20, and 30.

$A(0) = 0$

$A(20) = 30\sqrt{3}(20) - \sqrt{3}(20)^2 + \frac{\sqrt{3}}{4}(20)^2$

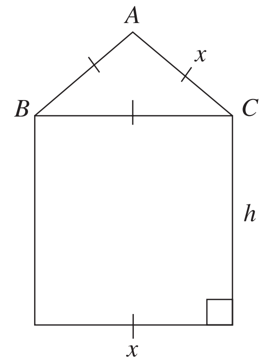
$= 520$

$A(30) = 30\sqrt{3}(30) - \sqrt{3}(30)^2 + \frac{\sqrt{3}}{4}(30)^2$

$\doteq 390$

The maximum area is 520 cm^2 when the base is 20 cm and each side is 20 cm.

10. a.



$4x + 2h = 6$

$2x + h = 3$ or $h = 3 - 2x$

Area $= xh + \frac{1}{2} \times x \times \frac{\sqrt{3}}{2} x$

$= x(3 - 2x) + \frac{\sqrt{3}x^2}{4}$

$A(x) = 3x - 2x^2 + \frac{\sqrt{3}}{4} x^2$

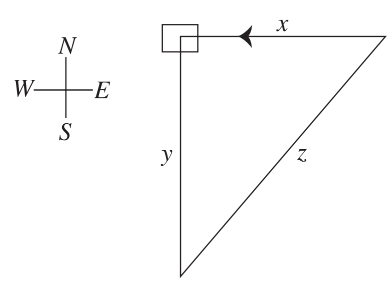
$A'(x) = 3 - 4x + \frac{\sqrt{3}}{2} x, 0 \le x \le 1.5$

For max or min, let $A'(x) = 0$, $x \doteq 1.04$.

$A(0) = 0$, $A(1.04) \doteq 1.43$, $A(1.5) \doteq 1.42$

The maximum area is approximately 1.43 cm^2 and occurs when $x = 0.96$ cm and $h = 1.09$ cm.

11.



Let z represent the distance between the two trains.

After t hours, $y = 60t$, $x = 45(1 - t)$

$z^2 = 3600t^2 + 45^2(1 - t)^2, 0 \le t \le 1$

$2z \frac{dz}{dt} = 7200t - 4050(1 - t)$

$\frac{dz}{dt} = \frac{7200t - 4050(1 - t)}{2\sqrt{3600t^2 + 45^2(1 - t)^2}}$

For max or min, let $\frac{dz}{dt} = 0$.

$\therefore\ 7200t - 4050(1 - t) = 0$

$t = 0.36$

When $t = 0$, $z^2 = 45^2$, $z = 45$

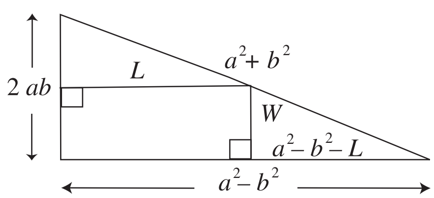
$t = 0.36$, $z^2 = 3600(0.36)^2 + 45^2(1 - 0.36)^2$

$z^2 = 129$

$z = 36$

$t = 1$, $z^2 = \sqrt{3600} = 60$

The closest distance between the trains is 36 km and occurs at 0.36 h after the first train left the station.

12. 

$$\frac{a^2 - b^2 - L}{a^2 - b^2} = \frac{W}{2ab}$$

$$W = \frac{2ab}{a^2 - b^2}(a^2 - b^2 - L)$$

$$A = LW = \frac{2ab}{a^2 - b^2}[a^2L - b^2L - L^2]$$

Let $\frac{dA}{dL} = a^2 - b^2 - 2L = 0$,

$$L = \frac{a^2 - b^2}{2}$$

and $W = \frac{2ab}{a^2 - b^2}\left[a^2 - b^2 - \frac{a^2 - b^2}{2}\right]$

$= ab$.

The hypothesis is proven.

13. Let the height be h and the radius r.

Then, $\pi r^2 h = k$, $h = \frac{k}{\pi r^2}$.

Let M represent the amount of material,

$M = 2\pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi r\left(\frac{k}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{2k}{r}$, $0 \leq r \leq \infty$

Using the max min Algorithm,

$$\frac{dM}{dr} = 4\pi r - \frac{2k}{r^2}$$

Let $\frac{dM}{dr} = 0$, $r^3 = \frac{k}{2\pi}$, $r \neq 0$ or $r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$.

When $r \to 0$, $M \to \infty$

$r \to \infty$, $M \to \infty$

$$r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$d = 2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$h = \frac{k}{\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}}} = \frac{k}{\pi} \bullet \frac{(2\pi)^{\frac{2}{3}}}{k^{\frac{2}{3}}} = \frac{k^{\frac{1}{3}}}{\pi^{\frac{1}{3}}} \bullet 2^{\frac{2}{3}}$$

Min amount of material is $M = 2\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}} + 2k\left(\frac{2\pi}{k}\right)^{\frac{1}{3}}$.

Ratio $\frac{h}{d} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \bullet 2^{\frac{2}{3}}}{2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \bullet 2^{\frac{2}{3}}}{2^{\frac{2}{3}}\left(\frac{k}{\pi}\right)^{\frac{1}{3}}} = \frac{1}{1}$

14. 

Cut the wire at P and label diagram as shown. Let AP form the circle and PB the square.

Then, $2\pi r = x$

$r = \frac{x}{2\pi}$

and the length of each side of the square is $\frac{100 - x}{4}$.

Area of circle $= \pi\left(\frac{x}{2\pi}\right)^2$

$= \frac{x^2}{4\pi}$

Area of square $= \left(\frac{100 - x}{4}\right)^2$

The total area is

$A(x) = \frac{x^2}{4\pi} + \left(\frac{100 - x}{4}\right)^2$, where $0 \leq x \leq 100$.

$A'(x) = \frac{2x}{4\pi} + 2\left(\frac{100 - x}{4}\right)\left(-\frac{1}{4}\right)$

$= \frac{x}{2\pi} - \frac{100 - x}{8}$

For max or min, let $A'(x) = 0$.

$$\frac{x}{2\pi} - \frac{100 - x}{8} = 0$$

$$x = \frac{100\pi}{r} + \pi \doteq 44$$

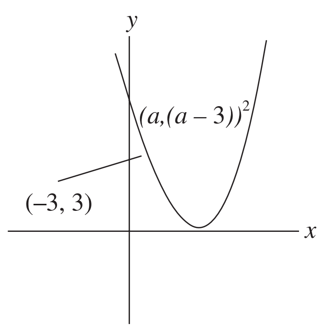
$$A(0) = 625$$

$$A(44) = \frac{44^2}{4\pi} + \left(\frac{100 - 44}{4}\right)^2 \doteq 350$$

$$A(100) = \frac{100^2}{4\pi} \doteq 796$$

The minimum area is 350 cm^2 when a piece of wire of approximately 44 cm is bent into a circle. The maximum area is 796 cm^2 and occurs when all of the wire is used to form a circle.

15.



Any point on the curve can be represented by $(a, (a - 3)^2)$.
The distance from $(-3, 3)$ to a point on the curve is
$d = \sqrt{(a + 3)^2 + ((a - 3)^2 - 3)^2}$.
To minimize the distance, we consider the function $d(a) = (a + 3)^2 + (a^2 - 6a + 6)^2$.
In minimizing $d(a)$, we minimize d since $d > 1$ always.
For critical points, set $d'(a) = 0$.
$d'(a) = 2(a + 3) + 2(a^2 - 6a + 6)(2a - 6)$
If $d'(a) = 0$,

$$a + 3 + (a^2 - 6a + 6)(2a - 6) = 0$$
$$2a^3 - 18a^2 + 49a - 33 = 0$$
$$(a - 1)(2a^2 - 16a + 33) = 0$$

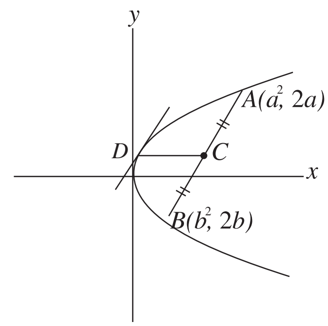
$$a = 1 \text{ or } a = \frac{16 \pm \sqrt{-8}}{4}$$

There is only one critical value, $a = 1$.
To determine whether $a = 1$ gives a minimal value, we use the second derivative test:

$$d'(a) = 6a^2 - 36a + 49$$
$$d''(1) = 6 - 36 + 49$$
$$\geq 0.$$

Then, $d(1) = 4^2 + 1^2$
$= 17$.
The minimal distance is $d = \sqrt{17}$, and the point on the curve giving this result is $(1, 4)$.

16.



Let the point A have coordinates $(a^2, 2a)$. (Note that the x-coordinate of any point on the curve is positive, but that the y-coordinate can be positive or negative. By letting the x-coordinate be a^2, we eliminate this concern.) Similarly, let B have coordinates $(b^2, 2b)$. The slope of AB is

$$\frac{2a - 2b}{a^2 - b^2} = \frac{2}{a + b}.$$

Using the mid-point property, C has coordinates

$$\left(\frac{a^2 + b^2}{2}, a + b\right).$$

Since CD is parallel to the x-axis, the y-coordinate of D is also $a + b$. The slope of the tangent at D is given by $\frac{dy}{dx}$ for the expression $y^2 = 4x$.

Differentiating,

$$2y\frac{dy}{dx} = 4$$

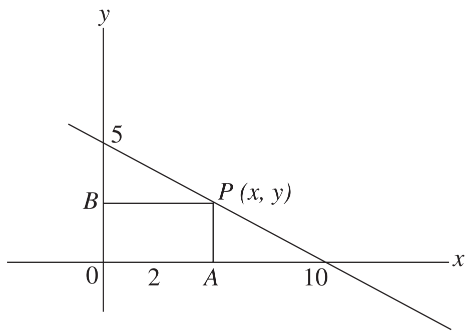
$$\frac{dy}{dx} = \frac{2}{y}$$

And since at point D, $y = a + b$,

$$\frac{dy}{dx} = \frac{2}{a + b}.$$

But this is the same as the slope of AB. Then, the tangent at D is parallel to the chord AB.

17.



Let the point $P(x, y)$ be on the line $x + 2y - 10 = 0$.

Area of $\Delta APB = xy$

$x + 2y = 10$ or $x = 10 - 2y$

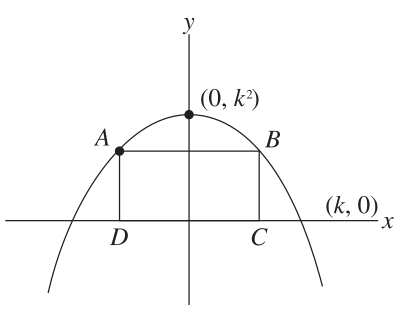
$$A(y) = (10 - 2y)y$$
$$= 10y - 2y^2,\ 0 \le y \le 5$$
$$A'(y) = 10.4y$$

For max or min, let $A'(y) = 0$ or $10 - 4y = 0$, $y = 2.5$,

$$A(0) = 0$$
$$A(2.5) = (10 - 5)(2.5) = 12.5$$
$$A(5) = 0.$$

The largest area is 12.5 units squared and occurs when P is at the point $(5, 2.5)$.

18.



A is $(-x, y)$ and $B(x, y)$

Area $= 2xy$ where $y = k^2 - x^2$

$$A(x) = 2x(k^2 - x^2)$$
$$= 2k^2x - 2x^3,\ -k \le x \le k$$
$$A'(x) = 2k^2 - 6x^2$$

For max or min, let $A'(x) = 0$,

$$6x^2 = 2k^2$$
$$x = \pm \frac{k}{\sqrt{3}}$$

When $x = \pm \frac{k}{\sqrt{3}}$, $y = k^2 - \left(\frac{k}{\sqrt{3}}\right)^2 = \frac{2}{3}k^2$

Max area is $A = \frac{2k}{\sqrt{3}} \times \frac{2}{3}k^2 = \frac{4k^3}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{4k^3}{9}$ square units.

Exercise 5.6

1. **a.** $C(625) = 75(\sqrt{625} - 10)$
$= 1125$

Average cost is $\frac{1125}{625} = \$1.80$.

b.
$$C(x) = 75(\sqrt{x} - 10)$$
$$= 75\sqrt{x} - 750$$
$$C'(x) = \frac{75}{2\sqrt{x}}$$
$$C'(1225) = \frac{75}{2\sqrt{1225}} = \$1.07$$

c. For a marginal cost of \$0.50/L,

$$\frac{75}{2\sqrt{x}} = 0.5$$
$$75 = \sqrt{x}$$
$$x = 5625$$

The amount of product is 5625 L.

3. $L(t) = \frac{6t}{t^2 + 2t + 1}$

a. $$L'(t) = \frac{6(t^2 + 2t + 1) - 6t(2t + 2)}{(t^2 + 2t + 1)^2}$$
$$= \frac{-6t^2 + 6}{(t^2 + 2t + 1)^2}$$

Let $L'(t) = 0$, then $-6t^2 + 6 = 0$,
$t^2 = 1$
$t = \pm 1$.

b. $L(1) = \frac{6}{1 + 2 + 1} = \frac{6}{4} = 1.5$

4. $C = 4000 + \frac{h}{15} + \frac{15\,000\,000}{h}$, $1000 \le h \le 20\,000$

$$\frac{dC}{dh} = \frac{1}{15} - \frac{15\,000\,000}{h^2}$$

Set $\frac{dC}{dh} = 0$, therefore, $\frac{1}{15} = \frac{15\,000\,000}{h^2} = 0$,

$$h^2 = 225\,000\,000$$
$$h = 15\,000,\ h > 0.$$

Using the max min Algorithm, $1000 \le h \le 20\,000$.

When $h = 1000$, $C = 4000 + \frac{1000}{15} + \frac{15\,000\,000}{1000}$,

$\doteq 19\,067$.

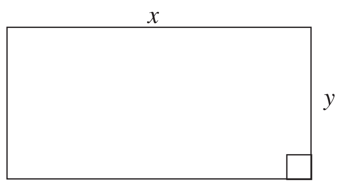
When $h = 15\,000$, $C = 4000 + \frac{15\,000}{15} + \frac{15\,000\,000}{15\,000}$,

$= 6000$.

When $h = 20\,000$, $C \doteq 6083$.

The minimum operating cost of \$6000/h occurs when the plane is flying at 15 000 m.

5.



Label diagram as shown and let the side of length x cost \$6/m and the side of length y be \$9/m.

Therefore, $(2x)(6) + (2y)(9) = 9000$

$2x + 3y = 1500$.

Area $A = xy$

But $y = \frac{1500 - 2x}{3}$.

$\therefore A(x) = x\left(\frac{1500 - 2x}{3}\right)$

$= 500x - \frac{2}{3}x^2$ for domain $0 \le x \le 500$

$A'(x) = 500 - \frac{4}{3}x$

Let $A'(x) = 0$, $x = 375$.

Using max min Algorithm, $0 \le x \le 500$,

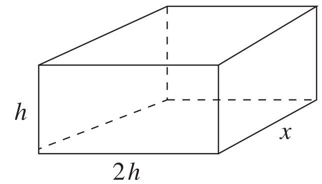
$A(0) = 0$, $A(375) = 500(375) - \frac{2}{3}(375)^2$

$= 93\,750$

$A(500) = 0$.

The largest area is 93 750 m^2 when the width is 250 m by 375 m.

6.



Label diagram as shown.

We know that $(x)(2h)(h) = 20\,000$

or $h^2x = 10\,000$

$x = \frac{10\,000}{h^2}$

Cost $C = 40(2hx) + 2xh(200) + 100(2)(2h^2 + xh)$

$= 80xh + 400xh + 400h^2 + 200xh$

$= 680xh + 400h^2$

Since $x = \frac{10\,000}{h^2}$,

$C(h) = 680h\left(\frac{10\,000}{h^2}\right) + 400h^2$, $0 \le h \le 100$

$C(h) = \frac{6\,800\,000}{h} + 400h^2$

$C'(h) = -\frac{6\,800\,000}{h^2} + 800h$.

Let $C'(h) = 0$,

$800h^3 = 6\,800\,000$

$h^3 = 8500$

$h \doteq 20.4$.

Apply max min Algorithm,

As $h \to 0$ $C(0) \to \infty$

$C(20.4) = \frac{6\,800\,000}{20.4} + 400(20.4)^2$

$= 499\,800$

$C(100) = 4\,063\,000$.

Therefore, the minimum cost is about \$500 000.

7. Let the height of the cylinder be h cm, the radius r cm. Let the cost for the walls be \$$k$ and for the top \$$2k$.

$V = 1000 = \pi r^2 h$ or $h = \dfrac{1000}{\pi r^2}$

The cost $C = (2\pi r^2)(2k) + (2\pi rh)k$

$$\text{or } C = 4\pi kr^2 + 2\pi kr\left(\frac{1000}{\pi r^2}\right)$$

$$C(r) = 4\pi kr^2 + \frac{2000k}{r},\ r \geq 0$$

$$C'(r) = 8\pi kr - \frac{2000k}{r^2}$$

Let $C'(r) = 0$, then $8\pi kr = \dfrac{2000k}{r^2}$

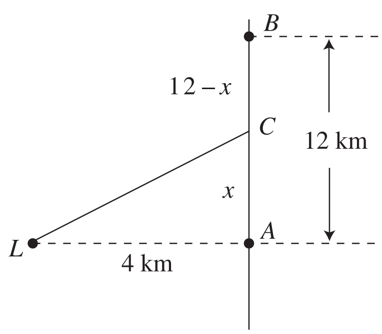
$$\text{or } r^3 = \frac{2000}{8\pi}$$

$$r \doteq 4.3$$

$$h = \frac{1000}{\pi(4.3)^2} = 17.2.$$

Since $r \geq 0$, minimum cost occurs when $r = 4.3$ cm and $h = 17.2$ cm.

8.



Let the distance AC be x km. Therefore, $CB = 12 - x$

$CL = \sqrt{16 + x^2}$.

Cost $C = 6000\sqrt{16 + x^2} + 2000(12 - x),\ 0 \leq x \leq 12$

$$\frac{dC}{dx} = \frac{1}{2} \times 6000 \times (16 + x^2)^{-\frac{1}{2}}(2x) + 2000(-1)$$

$$= \frac{6000x}{\sqrt{16 + x^2}} - 2000$$

Set $\dfrac{dC}{dx} = 0$

$$\frac{6000x}{\sqrt{16 + x^2}} = 2000$$

$$3x = \sqrt{16 + x^2}$$

$$9x^2 = 16 + x^2$$

$$x^2 = 2$$

$$x \doteq 1.4,\ x \geq 0$$

Using max min Algorithm:

when $x = 0$, $\quad C = 6000 \times 4 + 2000(12) = \$48\,000$

$x = 1.4$, $C = 6000 \times \sqrt{16 + (1.4)^2} + 2000(12 - 1.4)$
$\doteq \$46\,627$

$x = 12$, $C = 6000 \times \sqrt{16 + 12^2} \doteq 75\,895$.

The minimum cost occurs when point C is 1.4 km from point A or about 10.6 km south of the power plant.

9. Let the number of fare changes be x. Now, ticket price is \$20 + \$$0.5x$. The number of passengers is $10\,000 - 200x$.

The revenue $R(x) = (10\,000 - 200x)(20 + 0.5x)$,

$R'(x) = -200(20 + 0.5x) + 0.5(1000 - 200x)$

$\quad = -4000 - 100x + 5000 - 100x.$

Let $R'(x) = 0$:

$200x = 1000$

$x = 5.$

The new fare is \$20 + \$0.5(5) = \$22.50 and the maximum revenue is \$202 500.

10. Cost $C - \left(\dfrac{v^3}{2} + 216\right) \times t$

where $vt = 500$ or $t = \dfrac{500}{v}$.

$$C(v) = \left(\frac{v^3}{2} + 216\right)\left(\frac{500}{v}\right)$$

$$= 250v^2 + \frac{108\,000}{v},\ \text{where, } v \geq 0.$$

$$C'(v) = 500v - \frac{108\,000}{v^2}$$

Let $C'(v) = 0$, then $500v = \dfrac{108\,000}{v^2}$

$$v^3 = \frac{108\,000}{500}$$

$$v^3 = 216$$

$$v = 6.$$

The most economical speed is 6 nautical miles/h.

11. Let the number of increases be n.
New speed = $110 + n$.
Fuel consumption = $(8 - 0.1n)$ km/L.
For a 450 km trip:

fuel consumption = $\left(\frac{450}{8 - 0.1n}\right)$L,

fuel cost = $\left(\frac{450}{8 - 0.1n}\right)0.68$

Time for Trip = $\frac{D}{v} = \left(\frac{450}{110} + n\right)$ h

Cost = Cost of driver + fixed cost + fuel

$$C(n) = 35\left(\frac{450}{110 + n}\right) + 15.50\left(\frac{450}{110 + n}\right) + \left(\frac{450}{8 - 0.1n}\right)0.68$$

$$C'(n) = \frac{-15\,750}{(110 + n)^2} - \frac{6975}{(110 + n)^2} + \frac{30.6}{(8 - 0.1n)^2}$$

Let $C'(n) = 0$:

$$\frac{30.6}{(8 - 0.1n)^2} = \frac{22\,725}{(110 + n)^2}$$

$$\frac{(110 + n)^2}{(8 - 0.1n)^2} = \frac{22\,725}{30.6}$$

$$\frac{110 + n}{8 - 0.1n} = \pm\sqrt{742.6} = \pm 27.3$$

$110 + n = 27.3(8 - 0.1n)$
$n \doteq 29$
or $110 + n = -27.3(8 - 0.1n)$
$n \doteq 190.$
For $r \doteq 29$, new speed = 139 km/h
$n \doteq 190$, new speed = 300 km/h, which is not possible.
The speed is 139 km/h.

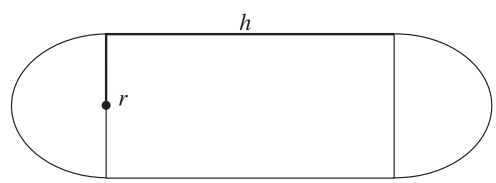
12. **a.** Let the number of \$0.50 increases be n.
New price = $10 + 0.5n$.
Number sold = $200 - 7n$.
Revenue $R(n) = (10 + 0.5n)(200 - 7n)$
$= 2000 + 30n - 3.5n^2$
Profit $P(n) = R(n) - C(n)$
$= 2000 + 30n - 3.5n^2 - 6(200 - 7n)$
$= 800 + 72n - 3.5n^2$
$P'(n) = 72 - 7n$
Let $P'(n) = 0$,
$72 - 7n = 0$, $n \doteq 10$.
Price per cake = 10 + 5 = \$15
Number sold = 200 – 70 = 130

b. Since 200 – 165 = 35, it takes 5 price increases to reduce sales to 165 cakes.
New price is $10 + 0.5 \times 5 = \$12.50$.

c. If you increase the price, the number sold will decrease. Profit in situations like this will increase for several price increases and then it will decrease because too many customers stop buying.

13. $P(x) = R(x) - C(x)$
Marginal Revenue = $R'(x)$.
Marginal Cost = $C'(x)$.
Now $P'(x) = R'(x) - C'(x)$.
The critical point occurs when $P'(x) = 0$.
If $R'(x) = C'(x)$ then $P'(x) = R'(x) - R'(x)$
$= 0$.
Therefore, the profit function has a critical point when the marginal revenue equals the marginal cost.

14.



Label diagram as shown. Let cost of cylinder be \$k/m^3.
$V = 200$
$= \pi r^2 h + \frac{4}{3}\pi r^3$
Note: Surface Area = Total cost C
Cost $C = (2\pi rh)k + (4\pi r^2)2k$
But, $200 = \pi r^2 h + \frac{4}{8}\pi r^3$ or $600 = 3\pi r^2 h + 4\pi r^2$
Therefore, $h = \frac{600 - 4\pi r^3}{3\pi r^2}$.

$$C(r) = 2k\pi r\left(\frac{600 - 4\pi r^3}{3\pi r^2}\right)1 + 8k\pi r^2$$

$$= 2k\left(\frac{600 - 4\pi r^3}{3r}\right) + 8k\pi r^2$$

Since $h \leq 16$, $r \leq \left(\frac{600}{4\pi}\right)^{\frac{1}{3}}$ or $0 \leq r \leq 3.6$

$$C(r) = \frac{400k}{r} - \frac{8k\pi r^2}{3} + 3k\pi r^2$$

$$= \frac{400k}{r} + \frac{16k\pi r^2}{3}$$

$$C'(r) = -\frac{400k}{r^2} + \frac{32k\pi r}{3}$$

Let $C'(r) = 0$

$$\frac{400k}{r^2} = \frac{32k\pi r}{3}$$

$$\frac{50}{r^2} = \frac{4\pi r}{3}$$

$$4\pi r^3 = 150$$

$$r^3 = \frac{150}{4\pi}$$

$$r = 2.29$$

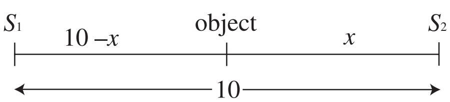
$$h \doteq 8.97 \text{ m}$$

Note: $C(0) \to \infty$

$C(2.3) \doteq 262.5k$

$C(3.6) \doteq 330.6k$

The minimum cost occurs when $r = 230$ cm and h is about 900 cm.

15. 

Note: $S_2 = 35_1$

Let x be the distance from S_2 to the object.

$I = \frac{ks}{x^2}$, where S is the strength of the source and x is the distance to the source.

$$I_1 = \frac{ks}{(10 - x)^2}$$

$$I_2 = \frac{k(35)}{x^2}, \ 0 < x < 10$$

$$I = \frac{ks}{(10 - x)^2} + \frac{3ks}{x^2}$$

$$\frac{dI}{dx} = \frac{-2ks}{(10 - x)^3} - \frac{6ks}{x^3}$$

Let $\frac{dI}{dx} = 0$. Therefore, $\frac{2ks}{(10 - x)^3} = \frac{6ks}{x^3}$:

$$x^3 = 3(10 - x)^3$$

$$x = \sqrt[3]{3}(10 - x)$$

$$x \doteq 1.44(10 - x)$$

$$2.4x = 14.4$$

$$x \doteq 5.9.$$

Minimum illumination occurs when $x = 5.9$ m.

16. $v(r) = Ar^2(r_0 - r), \ 0 \le \ r \le \ r$

$v(r) = Ar_0 r^2 - Ar^3$

$v'(r) = 2Ar_0 r - 3Ar^2$

Let $v'(r) = 0$:

$2Ar_0 r - 3Ar^2 = 0$

$2r_0 r - 3r^2 = 0$

$r(2r_0 - 3r) = 0$

$r = 0$ or $r = \frac{2r_0}{3}$.

$v(0) = 0$

$$v\left(\frac{2r_0}{3}\right) = A\left(\frac{4}{9}r_0^{\,2}\right)\left(r_0 - \frac{2r_0}{3}\right)$$

$$= \frac{4}{27}r_0 A$$

$A(r_0) = 0$

The maximum velocity of air occurs when radius is $\frac{2r_0}{3}$.

Review Exercise

1. **d.** $x^2y^{-3} + 3 = y$

$$2xy^{-3} - 3x^2y^{-4}\frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2xy^{-3}}{1 + 3x^2y^{-4}}$$

$$= \frac{\frac{2x}{y^3}}{1 + \frac{3x^2}{y^4}}$$

$$= \frac{\frac{2x}{y^3}}{\frac{y^4 + 3x^2}{y^4}}$$

$$= \frac{2xy}{3x^2y^4}$$

2. **b.** $(x^2 + y^2)^2 = 4x^2y$

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 8xy + 4x^2\frac{dy}{dx}$$

At (1, 1),

$$2(1 + 1)\left(2 + 2\frac{dy}{dx}\right) = 8 \times 1 \times 1 + 4(1)^2\frac{dy}{dx}$$

$$8 + 8\frac{dy}{dx} = 8 + 4\frac{dy}{dx}$$

$$\frac{dy}{dx} = 0.$$

3. $x^{-2}y^6 + 2y^{-2} - 6 = 0$

$$-2x^{-3}y^6 + 6x^{-2}y^5\frac{dy}{dx} - 4y^{-3}\frac{dy}{dx} = 0$$

At (0.5, 1):

$$-2(0.5)^{-3}(1)^6 + 6(0.5)^{-2}(1)^5\frac{dy}{dx} - 4(1)^{-3}\frac{dy}{dx} = 0$$

$$-16 + 24\frac{dy}{dx} - 4\frac{dy}{dx} = 0$$

$$20\frac{dy}{dx} = 16$$

$$\frac{dy}{dx} = \frac{4}{5}.$$

At (0.5, –1):

$$-2(0.5)^{-3}(-1)^6 + 6(0.5)^{-2}(-1)^5\frac{dy}{dx} - 4(-1)^{-3}\frac{dy}{dx} = 0$$

$$-16 - 24\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$

$$20\frac{dy}{dx} = -16$$

$$\frac{dy}{dx} = \frac{-4}{5}.$$

6. $3x^2 - y^2 = 7$

$$6x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{3y - 3x\frac{dy}{dx}}{y^2} = \frac{3y - 3x\left(\frac{3x}{y}\right)}{y^2} = \frac{3y^2 - 9x^2}{y^2}$$

But, $3y^2 - 9x^2 = -3(3x^2 - y^2)$
$= -3 \times 7$
$= -21.$

Therefore, $y'' = \frac{-21}{y^2}.$

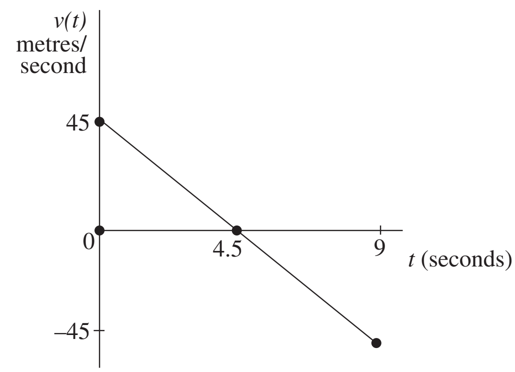
7. $s(t) = t^2 + (2t - 3)^{\frac{1}{2}}$

$$V = S'(t) = 2t + \frac{1}{2}(2t-3)^{-\frac{1}{2}}(2)$$
$$= 2t + (2t-3)^{-\frac{1}{2}}$$
$$a = S''(t) = 2 - \frac{1}{2}(2t-3)^{-\frac{3}{2}}(2)$$
$$= 2 - (2t-3)^{-\frac{3}{2}}$$

9. $s(t) = 45t - 5t^2$
$v(t) = 45 - 10t$
For $v(t) = 0$, $t = 4.5$.

t	$0 \le t < 4.5$	4.5	$t > 4.5$
$v(t)$	+	0	–

Therefore, the upward velocity is positive for $0 \le t < 4.5$ s, zero for $t = 4.5$ s, negative for $t > 4.5$ s.



10. a. $f(x) = 2x^3 - 9x^2$
$f'(x) = 6x^2 - 18x$
For max min, $f'(x) = 0$:
$6x(x - 3) = 0$
$x = 0$ or $x = 3$.

x	$f(x) = 2x^3 - 9x^2$	
–2	–52	min
0	0	max
3	–27	
4	–16	

The minimum value is –52.
The maximum value is 0.

c. $f(x) = 2x + \frac{18}{x}$

$f'(x) = 2 - 18x^{-2}$

For max min, $f'(x) = 0$:

$$\frac{18}{x^2} = 2$$

$$x^2 = 9$$

$$x = \pm 3.$$

x	$f(x) = 2x + \frac{18}{x}$
1	20
3	12
5	$10 + \frac{18}{5} = 13.6$

The minimum value is 12.
The maximum value is 20.

11. $s(t) = 62 - 16t + t^2$

$v(t) = -16 + 2t$

a. $s(0) = 62$

Therefore, the front of the car was 62 m from the stop sign.

b. When $v = 0$, $t = 8$.

$$\therefore s(8) = 62 - 16(8) + (8)^2 = 62 - 128 + 64 = -2$$

Yes, the car goes 2 m beyond the stop sign before stopping.

c. Stop signs are located two or more metres from an intersection. Since the car only went 2 m beyond the stop sign, it is unlikely the car would hit another vehicle travelling perpendicular.

12. $y^3 - 3xy - 5 = 0$

$$3y^2\frac{dy}{dx} - 3y - \frac{dy}{dx}(3x) = 0$$

At $(2, -1)$:

$$3\frac{dy}{dx} + 3 - 6\frac{dy}{dx} = 0$$

$$1 = \frac{dy}{dx}.$$

Equation of tangent at $(2, -1)$ is

$$\frac{y+1}{x-2} = 1$$

$$y = x - 3.$$

Therefore, the equation of the tangent at $(2, -1)$ to $y^3 - 3xy - 5 = 0$ is $y = x - 3$.

13. $s(t) = 1 + 2t - \frac{8}{t^2+1}$

$$v(t) = 2 + 8(t^2+1)^{-2}(2t) = 2 + \frac{16t}{(t^2+1)^2}$$

$$a(t) = 16(t^2+1)^{-2} + 16t(-2)(t^2+1)^{-3}\,2t$$
$$= 16(t^2+1)^{-2} - 64t^2(t^2+1)^{-3}$$
$$= 16(t^2+1)^{-3}[t^2 + 1 - 4t^2]$$

For max min velocities, $a(t) = 0$:

$$3t^2 = 1$$

$$t = \pm\frac{1}{\sqrt{6}}.$$

t	$v(t) = 2 + \frac{16t}{(t^2+1)^2}$
0	2 min
$\frac{1}{\sqrt{3}}$	$2 + \frac{\frac{16}{\sqrt{3}}}{\left(\frac{1}{3}+1\right)^2} = 2 + \frac{\frac{16\sqrt{3}}{3}}{\frac{16}{9}} = 2 + 3\sqrt{3}$ max
2	$2 + \frac{32}{25} = 3.28$

The minimum value is 2.
The maximum value is $2 + 3\sqrt{3}$.

14. $u(x) = 625x^{-1} + 15 + 0.01x$

$u'(x) = -625x^{-2} + 0.01$

For a minimum, $u'(x) = 0$

$$x^2 = 62\,500$$

$$x = 250$$

x	$u(x) = \frac{625}{x} + 0.01x$
1	625.01
250	2.5 + 2.5 = 5 min
500	$\frac{625}{500} + 5 = 6.25$

Therefore, 250 items should be manufactured to ensure unit waste is minimized.

15. **iii)** $C(x) = \sqrt{x} + 5000$

a. $C(400) = 20 + 5000$
$= \$5020$

b. $C(400) = \dfrac{5020}{400}$
$= \$12.55$

c. $C'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}$
$= \dfrac{1}{2\sqrt{x}}$

$C'(400) = \dfrac{1}{40}$
$= \$0.025$
$\doteq \$0.03$

$C'(401) = \dfrac{1}{2\sqrt{401}}$
$= \$0.025$
$\doteq \$0.03$

The cost to produce the 401st item is \$0.03.

iv) $C(x) = 100x^{-\frac{1}{2}} + 5x + 700$

a. $C(400) = \dfrac{100}{20} + 2000 + 700$
$= \$2705$

b. $C(400) = \dfrac{2750}{400}$
$= \$6.875$
$= \$6.88$

c. $C'(x) = -50x^{-\frac{3}{2}} + 5$

$C'(400) = \dfrac{-50}{(20)^3} + 5$
$= 5.00625$
$= \$5.01$

$C'(401) = \$5.01$

The cost to produce the 401st item is \$5.01.

16. $C(x) = 0.004x^2 + 40x + 16\ 000$

Average cost of producing x items is

$C(x) = \dfrac{C(x)}{x}$

$C(x) = 0.004x + 40 + \dfrac{16\ 000}{x}$

To find the minimum average cost, we solve

$$C'(x) = 0$$
$$0.004 - \frac{16\ 000}{x^2} = 0$$
$$4x^2 - 16\ 000\ 000 = 0$$
$$x^2 = 4\ 000\ 000$$
$$x = 2000,\ x > 0.$$

From the graph, it can be seen that $x = 2000$ is a minimum. Therefore, a production level of 2000 items minimizes the average cost.

17. **b.** $s(t) = -t^3 + 4t^2 - 10$

$s(0) = -10$

Therefore, its starting position is at -10.

$s(3) = -27 + 36 - 10$
$= -1$

$v(t) = -3t^2 + 8t$

$v(3) = -27 + 24$
$= -3$

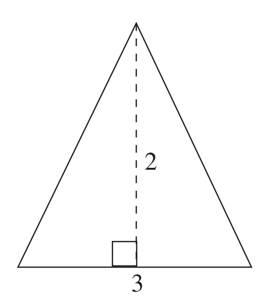
Since $s(3)$ and $v(3)$ are both negative, the object is moving away from the origin and towards its starting position.

18. Given cone $v = \dfrac{1}{3}\pi r^2 h$

$$\frac{dv}{dt} = 9\frac{m^3}{h}$$

Slopes of sides $= \dfrac{2}{3}$

$$\frac{rise}{run} = \frac{2}{3}$$
$$\therefore \frac{h}{r} = \frac{2}{3}$$



a. Find $\frac{dh}{dt}$ when $r = 6$ m.

Solution

$$v = \frac{1}{3}\pi r^2 h$$

Using $\frac{h}{r} = \frac{2}{3}$

$$r = \frac{3}{2h}$$

$$\therefore\ r = \frac{1}{3}\pi\left(\frac{9h^2}{4}\right)h$$

$$v = \frac{3}{4}\pi h^3$$

$$\frac{dv}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$$

At a specific time, $r = 6$ m:

$$\frac{h}{6} = \frac{2}{3}$$

$$h = 4 \text{ m.}$$

$$9 = \frac{9}{4}\pi(4)^2 \frac{dh}{dt}$$

$$9 = 36\pi \frac{dh}{dt}$$

$$\frac{1}{4\pi} = \frac{dh}{dt}$$

Therefore, the altitude is increasing at a rate of $\frac{1}{4\pi}$ m/h when $r = 6$ m.

b. Find $\frac{dr}{dt}$ when $h = 10$ m.

Solution

$$v = \frac{1}{3}\pi r^2 h$$

Using $\frac{h}{r} = \frac{2}{3}$,

$$h = \frac{2}{3}r$$

$$v = \frac{1}{3}\pi r^2\left(\frac{2}{3}r\right)$$

$$v = \frac{2}{9}\pi r^3$$

$$\frac{dv}{dt} = \frac{2}{3}\pi r^2 \frac{dr}{dt}$$

At a specific time, $h = 10$ m,

$$\frac{10}{r} = \frac{2}{3}$$

$$r = 15 \text{ m}$$

$$9 = \frac{2}{3}\pi(15)^2 \frac{dr}{dt}$$

$$9 = 150\pi \frac{dr}{dt}$$

$$\frac{3}{50\pi} = \frac{dr}{dt}.$$

Therefore, the radius is increasing at a rate of $\frac{3}{50\pi}$ m/h when $h = 10$ m.

19. Given $\frac{dv}{dt} = 1$ cm^3/s

Surface area = circular with $h = 0.5$ cm.

Volume is a cylinder.

Find $\frac{dA}{dt}$.

Solution

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$v = \pi r^2 h$$

But $h = 0.5$ cm,

$$v = \frac{1}{2}\pi r^2$$

$$\frac{dv}{dt} = \pi r \frac{dr}{dt}$$

At a specific time,

$$1 = \pi r \frac{dr}{dt}$$

$$\frac{1}{\pi r} = \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\left(\frac{1}{\pi r}\right)$$

$$= 2.$$

Therefore, the top surface area is increasing at a rate of 2 cm^2/s.

20. Given cube $s = 6x^2$

$$v = x^3$$

$$\frac{ds}{dt} = 8 \text{ cm}^2/\text{s},$$

find $\frac{dv}{dt}$ when $s = 60$ cm^2.

Solution

$v = x^2$

$\frac{dv}{dt} = 3x^2 \frac{dx}{dt}$

At a specific time, $s = 60$ cm^2,

$\therefore\ 6x^2 = 60$

$x^2 = 10.$

Also, $s = 6x^2$.

$\frac{ds}{dt} = 12x \frac{dx}{dt}$

At a specific time, $x = \sqrt{10}$,

$8 = 12\sqrt{10} \frac{dx}{dt}$

$\frac{2}{3\sqrt{10}} = \frac{dx}{dt}$

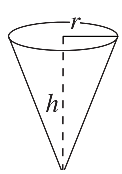
$\frac{dv}{dt} = 3(10)\left(\frac{2}{3\sqrt{10}}\right)$

$= \frac{20}{\sqrt{10}}$

$= 2\sqrt{10}.$

Therefore, the volume is increasing at a rate of $2\sqrt{10}$ cm^3/s.

21. Given cone



$v = \frac{1}{3}\pi r^2 h$

$\frac{dv}{dt} = -10$ cm^3/min

$\frac{dh}{dt} = -2$ cm/min

Find $\frac{h}{r}$ when $h = 8$ cm.

Solution

Let $\frac{h}{r} = k$, k is a constant $k \in R$

$v = \frac{1}{3}\pi r^2 h$ and $r = \frac{h}{k}$

$\therefore\ v = \frac{1}{3}\pi \frac{h^2}{k^2} h$

$v = \frac{1}{3k^2} \pi h^3$

$\frac{dv}{dt} = \frac{1}{k^2} \pi h^2 \frac{dh}{dt}.$

At a specific time, $h = 8$ cm,

$-10 = \frac{1}{k^2} \pi(8)^2(-2)$

$-10 = \frac{-128\pi}{k^2}$

$k^2 = \frac{64\pi}{5}$

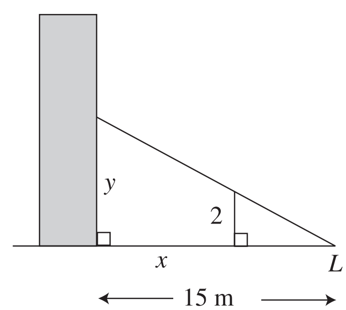
$k = \frac{8\sqrt{\pi}}{\sqrt{5}}$

$\therefore\ \frac{h}{r} = 8\sqrt{\frac{\pi}{5}}$

$= \frac{8\sqrt{5\pi}}{5}.$

Therefore, the ratio of the height to the radius is $8\sqrt{5\pi} : 5$.

22. Given



Since the angle formed from the light to the top of the man's head decreases as he walks towards the building, the length of his shadow on the building is decreasing.

Solution

Let x represent the distance he is from the wall. Therefore, $\frac{dx}{dt} = -2$, since he is walking towards the building. Let y be the length of his shadow on the building. Therefore, $\frac{dy}{dt}$ represents the rate of change of the length of his shadow.

Using similar triangles,

$$\frac{y}{15} = \frac{2}{15 - x}$$

$15y - xy = 30$

$15y - 30 = xy$

$$15\frac{dy}{dt} = \frac{dx}{dt}y + \frac{dy}{dt}x$$

At a specific time, $15 - x = 4$,

$\therefore x = 11$.

And using $\frac{y}{15} = \frac{2}{15 - x}$,

$$15\left(\frac{dy}{dt}\right) = (-2)(7.5) + \left(\frac{dy}{dt}\right)(11)$$

$$4\frac{dy}{dt} = -15$$

$$\frac{dy}{dt} = \frac{-15}{4}$$

Therefore, the length of his shadow is decreasing at a rate of 3.75 m/s.

23. $s = 27t^3 + \frac{16}{t} + 10, t > 0$

a. $v = 81t^2 - \frac{16}{t^2}$

$$81t^2 - \frac{16}{t^2} = 0$$

$81t^4 = 16$

$$t^4 = \frac{16}{81}$$

$$t = \pm\frac{2}{3}$$

$t > 0$

Therefore, $t = \frac{2}{3}$.

b.

t	$0 < t < \frac{2}{3}$	$t = \frac{2}{3}$	$t > \frac{2}{3}$
$\frac{ds}{dt}$	$-$	0	$+$

A minimum velocity occurs at $t = \frac{2}{3}$ or 0.67.

c. $a = \frac{dv}{dt} = 162t + \frac{32}{t^3}$

At $t = \frac{2}{3}$, $a = 162 \times \frac{2}{3} + \frac{32}{\frac{8}{27}}$

$= 216$

Since $a > 0$, the particle is accelerating.

24. Let the base be x cm by x cm and the height h cm.
Therefore, $x^2h = 10\,000$.

$A = x^2 + 4xh$

But $h = \frac{10\,000}{x^2}$,

$$A(x) = x^2 + 4x\left(\frac{10\,000}{x^2}\right)$$

$$= x^2 + \frac{400\,000}{x}, \text{ for } x \geq 5$$

$$A'(x) = 2x - \frac{400\,000}{x^2},$$

Let $A'(x) = 0$, then $2x = \frac{400\,000}{x^2}$

$x^3 = 200\,000$

$x = 27.14$.

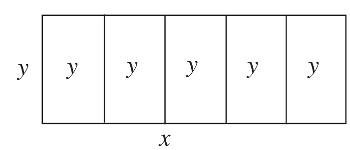
Using the max min Algorithm,

$A(5) = 25 + 80\,000 = 80\,025$

$A(27.14) \doteq 15\,475$

The dimensions of a box of minimum area is 27.14 cm for the base and height 13.57 cm.

25. Let the length be x and the width y.



$$P = 2x + 6y \text{ and } xy = 12\,000 \text{ or } y = \frac{12\,000}{x}$$

$$P(x) = 2x + 6 \times \frac{12\,000}{x}$$

$$P(x) = 2x + \frac{72\,000}{x}, 10 \leq x \leq 1200\ (5 \times 240)$$

$$A'(x) = 2 - \frac{72\,000}{x^2}$$

Let $A'(x) = 0$,

$2x^2 = 72\,000$

$x^2 = 36\,000$

$x \doteq 190$.

Using max min Algorithm,

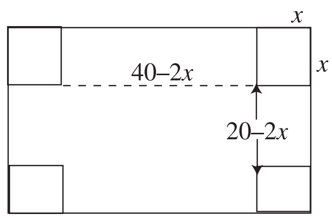
$A(10) = 20 + 7200 = 7220$ m^2

$A(190) \doteq 759$ m^2

$A(1200) = 1\,440\,060$

The dimensions for the minimum amount of fencing is a length of 190 m by a width of approximately 63 m.

26.



Let the width be w and the length $2w$.

Then, $2w^2 = 800$

$w^2 = 400$

$w = 20,\ w > 0.$

Let the corner cuts be x cm by x cm. The dimensions of the box are shown. The volume is

$V(x) = x(40 - 2x)(20 - 2x)$

$= 4x^3 - 120x^2 - 800x,\ 0 \leq x \leq 10$

$V'(x) = 12x^2 - 240x - 800$

Let $V'(x) = 0$:

$12x^2 - 240x - 800 = 0$

$3x^2 - 60x - 200 = 0$

$$x = \frac{60 \pm \sqrt{3600 - 2400}}{6}$$

$x \doteq 15.8$ or $x = 4.2$, but $x \leq 10$.

Using max min Algorithm,

$V(0) = 0$

$V(4.2) = 1540\ \text{cm}^2$

$V(10) = 0.$

Therefore, the base is

$40 - 2 \times 4.2 = 31.6$

by $20 - 2 \times 4.2 = 11.6$.

The dimensions are 31.6 dm^2, by 11.6 dm, by 4.2 dm.

27. Let the radius be r cm and the height h cm.

$V = \pi r^2 h = 500$

$A = 2\pi r^2 + 2\pi rh$

Since $h = \dfrac{500}{\pi r^2},\ 6 \leq h \leq 15$

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{1000}{r} \text{ for } 2 \leq r \leq 5$$

$$A'(r) = 4\pi r - \frac{1000}{r^2}.$$

Let $A'(r) = 0$, then $4\pi r^3 = 1000$,

$$r^3 = \frac{1000}{4\pi}$$

$$r \doteq 4.3.$$

Using max min Algorithm,

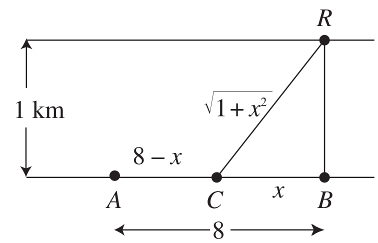
$A(2) \doteq 550$

$A(4.3) \doteq 349$

$A(5) \doteq 357$

For a minimum amount of material, the can should be constructed with a radius of 4.3 cm and a height of 8.6 cm.

28.



Let x be the distance CB, and $8 - x$ the distance AC.

Let the cost on land be $\$k$ and under water $\$1.6k$.

The cost $C(x)k(8 - x) + 1.6k\sqrt{1 + x^2},\ 0 \leq x \leq 8.$

$$C'(x) = -k + 1.6k \times \frac{1}{2}(1 + x^2)^{-\frac{1}{2}}(2x)$$

$$= -k + \frac{1.6kx}{\sqrt{1 + x^2}}$$

Let $C'(x) = 0$,

$$-k + \frac{1.6kx}{\sqrt{1 + k^2}} = 0$$

$$\frac{1.6x}{\sqrt{1 + k^2}} = 1$$

$1.6x = \sqrt{1 + x^2}$

$2.56x^2 = 1 + x^2$

$1.56x^2 = 1$

$x^2 \doteq 0.64$

$x = 0.8,\ x > 0.$

Using max min Algorithm,

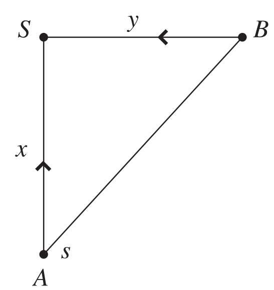
$A(0) = 9.6k$

$A(0.8) = k(8 - 0.8) + 1.6k\sqrt{1 + (0.8)^2} = 9.25k$

$A(8) = 12.9k$

The best way to cross the river is to run the pipe 8 – 0.8 or 7.2 km along the river shore and then cross diagonally to the refinery.

29.



Let y represent the distance the westbound train is from the station and x the distance of the northbound train from the station S. Let t represent time after 10:00.

Then $x = 100t$, $y = (120 - 120t)$

Let the distance AB be z.

$$z = \sqrt{(100t)^2 + (120 - 120t)^2},\ 0 \le t \le 1$$

$$\frac{dz}{dt} = \frac{1}{2}\left[(100t)^2 + (120 - 120t)^2\right]^{-\frac{1}{2}}$$
$$\left[2 \times 100 \times 100t - 2 \times 120 \times (120(1 - t)\right]$$

Let $\frac{dz}{dt} = 0$, that is

$$\frac{2 \times 100 \times 100t - 2 \times 120 \times 120(1 - t)}{2\sqrt{(100t)^2 + (120 - 120t)^2}} = 0$$

or $20\,000t = 28\,800(1 - t)$

$48\,800t = 288\,000$

$$t = \frac{288}{488} \doteq 0.59 \text{ h or } 35.4 \text{ min.}$$

When $t = 0$, $z = 120$.

$t = 0.59$

$z = \sqrt{(100 \times 0.59)^2 + (120 - 120 \times 0.59)^2}$

$= 76.8$ km

$t = 1$, $z = 100$

The closest distance between trains is 76.8 km and occurs at 10:35.

30. Let the number of price increases be n.

New selling price $= 100 + 2n$.

Number sold $= 120 - n$.

Profit = Revenue – Cost

$P(n) = (100 + 2n)(120 - n) - 70(120 - n),\ 0 \le n \le 120$

$= 3600 + 210n - 2n^2$

$P'(n) = 210 - 4n$

Let $P'(n) = 0$

$210 - 4n = 0$

$n = 52.5.$

Therefore, $n = 52$ or 53.

Using max min Algorithm,

$P(0) = 3600$

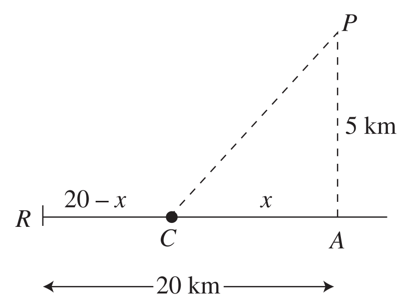
$P(52) = 9112$

$P(53) = 9112$

$P(120) = 0$

The maximum profit occurs when the CD players are sold at \$204 for 68 and at \$206 for 67 CD players.

31.



Let x represent the distance AC.

Then, $RC = 20 - x$ and 4.

$PC = \sqrt{25 + x^2}$

The cost:

$C(x) = 100\,000\sqrt{25 + x^2} + 75\,000(20 - x),\ 0 \le x \le 20$

$C'(x) = 100\,000 \times \frac{1}{2}\left(25 + x^2\right)^{-\frac{1}{2}}(2x) - 75\,000.$

Let $C'(x) = 0$,

$$\frac{100\,000x}{\sqrt{25 + x^2}} - 75\,000 = 0$$

$4x = 3\sqrt{25 + x^2}$

$16x^2 = 9(25 + x^2)$

$7x^2 = 225$

$x^2 \doteq 32$

$x \doteq 5.7.$

Using max min Algorithm,

$A(0) = 100\,000\sqrt{25} + 75\,000\,(20) = 2\,000\,000$

$A(5.7) = 100\,000\sqrt{25 + 5.7^2} + 75\,000(20 - 5.7)$

$= 1\,830\,721.60$

$A(20) = 2\,061\,552.81.$

The minimum cost is \$1 830 722 and occurs when the pipeline meets the shore at a point C, 5.7 km from point A, directly across from P.

Chapter 5 Test

1. $x^2 + 4xy - y^2 = 8$

$$2x + 4y + 4x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$x + 2y + \frac{dy}{dx}(2x - y) = 0$$

$$\frac{dy}{dx} = \frac{-x - 2y}{2x - y}$$

$$\frac{dy}{dx} = \frac{x + 2y}{y - 2x}$$

2. $3x^2 + 4y^2 = 7$

$$6x + 8y\frac{dy}{dx} = 0$$

At $P(-1, 1)$,

$$-6 + 8\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3}{4}.$$

Equation of tangent line at $P(-1, 1)$ is

$$\frac{y - 1}{x + 1} = \frac{3}{4}$$

$$3x + 3 = 4y - 4$$

$$3x - 4y + 7 = 0.$$

3. **a.** Average velocity from $t = 1$ to $t = 6$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(6) - s(1)}{6 - 1}$$

$$= \frac{(216 - 324 + 144 + 5) - (1 - 9 + 24 + 5)}{5}$$

$$= 4 \text{ m/s}.$$

The average velocity from $t = 1$ to $t = 6$ is 4 m/s.

b. Object is at rest when $v = 0$:

$$0 = 3t - 18t + 24$$

$$= 3(t^2 - 6t + 8)$$

$$= 3(t - 4)(t - 2)$$

$$t = 2 \text{ or } t = 4.$$

Therefore, the object is at rest at 2 s and 4 s.

c. $v(t) = 3t^2 - 18t + 24$

$a(t) = 6t - 18$

$a(5) = 30 - 18$

$= 12$

Therefore, the acceleration after 5 s is 12 m/s^2.

d. $s(3) = 27 - 81 + 72 + 5$

$= 23$

$v(3) = 27 - 54 + 24$

$= -3$

Since the signs of $s(3)$ and $v(3)$ are different, the object is moving towards the origin.

4. Given: circle $\frac{dr}{dt} = 2$ m/s,

find $\frac{dA}{dt}$ when $r = 60$.

Solution

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

At a specific time, $r = 60$,

$$\frac{dA}{dt} = 2\pi(60)(2)$$

$$= 240\pi.$$

Therefore, the area is increasing at a rate of 240π m^2/s.

5. Given: sphere $\frac{dr}{dt} = 2$ m/min,

find $\frac{dv}{dt}$ when $r = 8$ m.

Solution

$$v = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}.$$

At a specific time, $r = 8$ m:

$$\frac{dv}{dt} = 4\pi(64)(2)$$

$$= 512\pi.$$

a. Therefore the volume is increasing at a rate of 512π m^3/min.

b. The radius is increasing, therefore the volume is also increasing. Answers may vary.

6. Given: cube $V = x^3$, $S = 6x^2$, $\frac{dV}{dt} = 2$ cm^3/min,

find $\frac{dS}{dt}$ when $x = 5$.

Solution

$S = 6x^2$

$\frac{dS}{dt} = 12x\,\frac{dx}{dt}$

At a specific time, $x = 5$:

$\frac{dS}{dt} = 12(5)\left(\frac{2}{75}\right)$

$= \frac{8}{5}$

$V = x^3$

$\frac{dV}{dt} = 3x^2\,\frac{dx}{dt}$

When $x = 5$, $\frac{dV}{dt} = 2$

$2 = 3(5)^2\,\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{2}{75}$

$S = 6x^2$

$\frac{dS}{dt} = 12x\,\frac{dx}{dt}$

When $x = 5$, $\frac{dx}{dt} = \frac{2}{75}$.

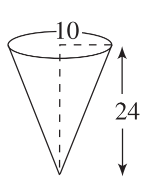
Therefore, $\frac{dS}{dt} = 12(5)\left(\frac{2}{75}\right)$

$= \frac{8}{5}$

$= 1.6.$

Therefore, the surface area of the cube is increasing at a rate of 1.6 cm^2/min.

7.



$\frac{dv}{dt} = 20$ m^3/min

Find $\frac{dh}{dt}$ when $h = 16$.

Solution

$v = \frac{1}{3}\pi r^2 h$

$\frac{r}{h} = \frac{10}{24} = \frac{5}{12}$

$v = \frac{5}{12}h.$

Substituting into v,

$v = \frac{1}{3}\pi\left(\frac{25}{144}h^2\right)h$

$v = \frac{25}{432}\pi h^3.$

$\frac{dv}{dt} = \frac{25}{144}\pi h^2\,\frac{dh}{dt}$

At a specific time, $h = 16$:

$20 = \frac{25}{144}\pi(16)^2\,\frac{dh}{dt}$

$20 = \frac{6400\pi}{144}\,\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{9}{20\pi}.$

Therefore, the depth of the water is increasing at a rate of $\frac{9}{20\pi}$ m/min.

8. $f(x) = \frac{x^2 - 1}{x + 2}$

$f'(x) = \frac{2x(x + 2) - (x^2 - 1)(1)}{(x + 2)^2}$

$= \frac{x^2 + 4x + 1}{(x + 2)^2}$

For max min, $f'(x) = 0$:

$x^2 + 4x + 1 = 0$

$x = \frac{-4 \pm \sqrt{16 - 4}}{2}$

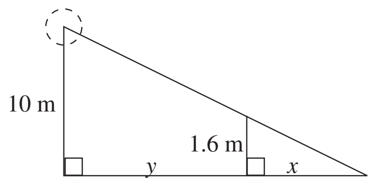
$= \frac{-4 \pm 2\sqrt{3}}{2}$

$= -2 \pm \sqrt{3}.$

x	$f(x) = \frac{x^2 - 2}{x + 2}$
-1	0
$-2 + \sqrt{3}$	$\frac{4 - 4\sqrt{3} + 3 - 1}{\sqrt{3}} = \frac{6 - 4\sqrt{3}}{\sqrt{3}} = 2\sqrt{3} - 4$ $\doteq -0.536$ min
3	$\frac{8}{5} = 1.6$ max $\sqrt{3}$

Therefore, the minimum value is $(2\sqrt{3} - 4)$ and the maximum value is 1.6.

9.



Find $\frac{dx}{dt}$ when $y = 8$ m.

Let x represent the distance the tip of her shadow is from the point directly beneath the spotlight.

Let y represent the distance she is from the point directly beneath the spotlight.

$\frac{dy}{dx} = 6$ m/s

Solution

$$\frac{x}{x - y} = \frac{10}{1.6} = 6.25$$

$$x = 6.25x - 6.25y$$

$$6.25y = 5.25x$$

$$6.25\frac{dy}{dt} = 5.25\frac{dx}{dt}$$

At a specific time, $y = 8$ m:

$$6.25(6) = 5.25\frac{dx}{dt}$$

$$7.1 \doteq \frac{dx}{dt} \quad \text{or} \quad = 21\frac{dx}{dt}.$$

At a specific time,

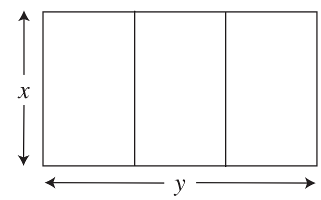
$$25(6) = 2\frac{dx}{dt}$$

$$\frac{150}{21} = \frac{dx}{dt}$$

$$\frac{50}{7} = \frac{dx}{dt}.$$

Therefore, her shadow's head is moving away at a rate of 7.1 m/s.

10.



Let x represent the width of the field in m, $x > 0$.

Let y represent the length of the field in m.

$4x + 2y = 2000$ (1)

$A = xy$ (2)

From (1): $y = 1000 - 2x$. Restriction $0 < x < 500$

Substitute into (2):

$$A(x) = x(1000 - 2x)$$

$$= 1000x - 2x^2$$

$$A'(x) = 1000 - 4x.$$

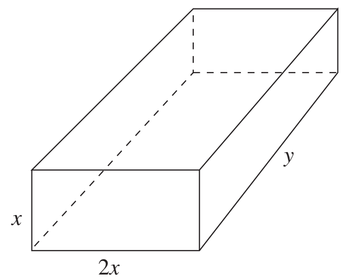
For a max min, $A'(x) = 0$, $x = 250$

x	$A(x) = x(1000 - 2x)$
0	$\lim_{x \to 0^+} A(x) = 0$
250	$A(250) = 125\ 000$ max
1000	$\lim_{x \to 1000} A(x) = 0$

$\therefore$ $x = 250$ and $y = 500$.

Therefore, each paddock is 250 m in width and $\frac{500}{3}$ m in length.

11.



Let x represent the height.
Let $2x$ represent the width.
Let y represent the length.
Volume $10\ 000 = 2x^2y$
Cost:

$$\begin{aligned} C &= 0.02(2x)y + 2(0.05)(2x^2) + 2(0.05)(xy) + 0.1(2xy) \\ &= 0.04xy + 0.2x^2 + 0.1xy + 0.2\ xy \\ &= 0.34xy + 0.2x^2 \end{aligned}$$

But $y = \dfrac{10\ 000}{2x^2} = \dfrac{5000}{x^2}$.

$$\text{Therefore, } C(x) = 0.34x\left(\frac{5000}{x^2}\right) + 0.2x^2$$

$$= \frac{1700}{x} + 0.2x^2,\ x \geq 0$$

$$C'(x) = \frac{-1700}{x^2} + 0.4x.$$

Let $C'(x) = 0$:

$$\frac{-1700}{x^2} + 0.4x = 0$$

$$0.4x^3 = 1700$$

$$x^3 = 4250$$

$$x \doteq 16.2.$$

Using max min Algorithm,

$C(0) \to \infty$

$$C(16.2) = \frac{1700}{16.2} + 0.2(16.2)^2 = 157.4.$$

Minimum when $x = 16.2$, $2x = 32.4$ and $y = 19.0$.
The required dimensions are 162 m, 324 m by 190 m.